p-divisible groups

J.M. Commelin

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1 Notation. — In these notes, instead of $\varinjlim$ we write colim, and instead of $\varprojlim$ we write lim.

2 — Let K be a number field. Let A be an abelian variety over K. Let ℓ be a prime number. Let $A_{\ell^\infty}(\bar{K}) = \bigcup_n A[\ell^n](\bar{K})$ denote the ℓ-divisible subgroup of $A(\bar{K})$. Let W be a $\mathrm{Gal}(\bar{K}/K)$-stable subgroup of $A_{\ell^\infty}(\bar{K})$. For every n, let $B(n)$ denote the quotient A/W_{ℓ^n}.

We want to bound the height of $B(n)$. More precisely, we want the following result.

3 Theorem. — *For $n \gg 0$, the height $h(B(n))$ does not depend on n.*
In this talk I will not prove this theorem. Rather, I will present an overview of some facts about p-divisble groups, and in the end prove a proposition that will be very useful for proving theorem 3.

4 — We will use [1] as main reference.

Let R be a ring (or scheme). Let p be a prime number. Let h be an integer ≥ 0. By definition, a group scheme over R has rank h if it is locally free of rank h over R (in other words, it is defined by a Hopf algebra that is locally free of rank h over R).

5 Definition. — A *p-divisible group* of height h is an inductive system

$$G = (G_\nu, i_\nu)_{\nu \geq 0}$$

where G_ν is a finite group scheme over R of order $p^{\nu h}$, and such that for each $\nu \geq 0$, the sequence

$$0 \longrightarrow G_\nu \xrightarrow{i_\nu} G_{\nu+1} \xrightarrow{[p^\nu]} G_{\nu+1}$$

is exact. (So G_ν is the set of p^ν-torsion points in $G_{\nu+1}$.)

A homomorphism of p-divisible groups is what you think it is.

Probably the best known example of a p-divisible group is given by

$$G_\nu = (\mathbb{Z}/p^\nu\mathbb{Z})^h \quad \text{and} \quad G = \mathrm{colim}\, G_\nu = (\mathbb{Q}_p/\mathbb{Z}_p)^h.$$

The next best known example is $A_{p^\infty}(\bar{K}) = \mathrm{colim}\, A[\ell^\nu](\bar{K})$, where A is an abelian variety over a field K, as in §2.

6 Consequences of the definition. — Let G be a p-divisible group. By iteration, we obtain closed immersions $i_{\nu,\mu}\colon G_\nu \to G_{\nu+\mu}$, for all $\nu, \mu \geq 0$. (Note that $i_{\nu,1} = i_\nu$.) These maps $i_{\nu,\mu}$ identify G_ν with the kernel of $[p^\nu]$ in $G_{\nu+\mu}$.

Consider the following diagram, with exact row and column.

$$
\begin{array}{ccccccc}
 & & & & & & G_{\nu+\mu} \\
 & & & & & & \uparrow{\scriptstyle [p^\nu]} \\
0 & \longrightarrow & G_\mu & \xrightarrow{i_{\mu,\nu}} & G_{\nu+\mu} & \xrightarrow{[p^\mu]} & G_{\nu+\mu} \\
 & & & & & & \uparrow{\scriptstyle i_{\nu,\mu}} \\
 & & & & & & G_\nu \\
 & & & & & & \uparrow \\
 & & & & & & 0
\end{array}
$$

Since the composition $[p^\nu] \circ [p^\mu] = [p^{\nu+\mu}]$ is identically 0 on $G_{\nu+\mu}$, we see that $[p^\mu]$ factors via a map $j_{\nu,\mu}\colon G_{\nu+\mu} \to G_\nu$.

$$
\begin{array}{ccccccc}
 & & & & & & G_{\nu+\mu} \\
 & & & & & \overset{0}{\nearrow} & \uparrow{\scriptstyle [p^\nu]} \\
0 & \longrightarrow & G_\mu & \xrightarrow{i_{\mu,\nu}} & G_{\nu+\mu} & \xrightarrow{[p^\mu]} & G_{\nu+\mu} \\
 & & & & & \underset{j_{\mu,\nu}}{\searrow} & \uparrow{\scriptstyle i_{\nu,\mu}} \\
 & & & & & & G_\nu \\
 & & & & & & \uparrow \\
 & & & & & & 0
\end{array}
$$

Observe that $i_{\nu,\mu} \circ j_{\mu,\nu} = [p^\mu]$. Because $i_{\nu,\mu}$ is an immersion, the sequence

$$0 \longrightarrow G_\mu \xrightarrow{i_{\mu,\nu}} G_{\nu+\mu} \xrightarrow{j_{\mu,\nu}} G_\nu$$

is exact. In fact, since the order of G_μ and G_ν add up to the order of $G_{\nu+\mu}$, we find that the last map is in fact a quotient map, and we obtain the short exact sequence

$$0 \longrightarrow G_\mu \xrightarrow{i_{\mu,\nu}} G_{\nu+\mu} \xrightarrow{j_{\mu,\nu}} G_\nu \longrightarrow 0.$$

We will write j_ν for $j_{1,\nu}$.

7 Tate modules. — Let R be an integral domain, with field of fractions K. Assume $\operatorname{char} K = 0$ and let $\bar{K}$ be an algebraic closure of K. Let G be a p-divisible group over R of height h. The *Tate module* of G is denoted $T(G)$, and is by definition $\lim G_\nu(\bar{K})$, where limit is taken over the morphisms j_ν. Dually, one defines $\Phi(G)$ as $\operatorname{colim} G_\nu(\bar{K})$, where the colimit is over the maps i_ν. *N.b.:* There is a notion of "points of G" which we do not need for the main result of this talk. It coincides with $\Phi(G)$ when G is étale, but contains $\Phi(G)$ as torsion subgroup in the general situation.

Since K has characteristic 0, the groups $G_\nu \otimes K$ are étale, and hence $T(G)$ is isomorphic as $\mathbb{Z}_p$-module to $\mathbb{Z}_p^h$, while $\Phi(G)$ is isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^h$. Furthermore, there is a continuous action of $\mathrm{Gal}(\bar{K}/K)$ on $T(G)$ and $\Phi(G)$. There are canonical isomorphisms (of Galois modules)

$$\Phi(G) \cong T(G) \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p) \quad \text{and} \quad T(G) \cong \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \Phi(G)).$$

Observe that one can recover the Galois module $G_\nu(\bar{K})$ from $\Phi(G)$ by taking the kernel of $[p^\nu]$. Using the well-known fact that a finite étale group scheme over a field is determined by its Galois module of $\bar{K}$-points, we may thus recover the generic fibre $G \otimes_R K$ from $\Phi(G)$ or $T(G)$.

8 Corollary. — *The assignment $G \mapsto T(G)$ establishes an equivalence of categories between the category of p-divisible groups over K and free $\mathbb{Z}_p$-modules of finite rank with a continuous action of* $\mathrm{Gal}(\bar{K}/K)$.

9 Proposition (prp. 12 of [1]). — *Let R be an integrally closed, Noetherian, integral domain, with field of fractions K. Fix a prime number p. Let G be a p-divisible group over R. Let $T(G)$ be the Tate module of G. Let W be a direct summand of $T(G)$ over $\mathbb{Z}_p$ that is stable under the action of* $\mathrm{Gal}(\bar{K}/K)$. *Then there exists a p-divisible group Γ over R, and a morphism $\phi\colon \Gamma \to G$ such that ϕ induces an isomorphism $T(\Gamma) \cong W$.*

Proof. By corollary 8 we immediately obtain a p-divisible subgroup $H_* \subset G \otimes K$. We want to take the closure H of H_* in G. To make this precise, let B_ν be the R-algebra corresponding to G_ν. Let $A_{*\nu}$ be the K-algebra corresponding to $H_{*\nu}$, and consider $u_\nu\colon B_\nu \otimes_R K \to A_{*\nu}$ corresponding to $H_{*\nu} \hookrightarrow G_\nu \otimes K$. Let A_ν be the image $u_\nu(B_\nu)$ and put $H_\nu = \mathrm{Spec}(A_\nu)$. Observe that A_ν is a cocommutative Hopf algebra, and therefore H_ν is a commutative group scheme.

$$\begin{array}{ccccc}
B_\nu \otimes_R K & \longrightarrow & A_\nu & \xrightarrow{\ \iota\ } & A_{*\nu} \\
\uparrow & & & & \uparrow \\
B_{\nu+1} \otimes_R K & \xrightarrow{\ \sigma\ } & A_{\nu+1} & \longrightarrow & A_{*\nu+1}
\end{array}$$

By construction ι is injective, while σ is surjective. Hence we obtain a map

$$\begin{array}{ccccc}
B_\nu \otimes_R K & \longrightarrow & A_\nu & \xrightarrow{\ \iota\ } & A_{*\nu} \\
\uparrow & & \vdots\uparrow & & \uparrow \\
B_{\nu+1} \otimes_R K & \xrightarrow{\ \sigma\ } & A_{\nu+1} & \longrightarrow & A_{*\nu+1}
\end{array}$$

and thus maps $H_\nu \to H_{\nu+1}$. Nevertheless, H is not necessarily a p-divisible group. (The last lines of [1] provide an example by Serre, that illustrates this problem.) However, $H \otimes K \cong H_*$ is a p-divisible group. As we will see, the failure of H being a p-divisible group is somehow only at a finite level. What I mean is this: for $\nu \gg 0$ we will see that $H_\nu \to H_{\nu+1}$ satisfies the axioms for a p-divisible group. We will exploit this to define Γ in terms of H.

Because all groups involved are finite, quotients such as $H_{\mu+1}/H_\mu$ exist. By looking at the generic fibre, we see that $H_{\mu+1}/H_\mu$ is killed by p. In particular the map $[p]$ induces maps

$$H_{\mu+\nu+1}/H_{\mu+1} \to H_{\mu+\nu}/H_\mu$$

that are isomorphisms on the generic fibre. (After all, on the generic fibre both the source and the target are isomorphic to H_ν, and the kernel of the map is 0.) Let D_μ be the algebra corresponding to $H_{\mu+1}/H_\mu$. By the above observation, the algebra $D_\mu \otimes_R K$ does not depend on μ; and the D_μ form an increasing sequence of orders inside a finite separable K-algebra.

From some point onwards, say μ_0, this sequence stabilises: $D_\mu = D_{\mu_0}$ for $\mu \geq \mu_0$. Now we may put $\Gamma_\nu = H_{\nu+\mu_0}/H_{\mu_0}$. Note that $[p^{\mu_0}]$ induces maps $\Gamma_\nu \to H_\nu$ that are isomorphisms on the generic fibre. Hence (if we assume for a moment that Γ is p-divisible), it is immediate that $\Gamma \hookrightarrow G$ induces an isomorphism $T(\Gamma) \to W$.

We are done if we show that Γ is p-divisible. To see this, consider the following diagram.

$$\begin{array}{ccc} H_{\nu+\mu_0+1}/H_{\mu_0} & \xrightarrow{[p^\nu]} & H_{\nu+\mu_0+1}/H_{\mu_0} \\ \downarrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\gamma} \\ H_{\nu+\mu_0+1}/H_{\nu+\mu_0} & \xrightarrow{\beta} & H_{\mu_0+1}/H_{\mu_0} \end{array}$$

Here

- » α is the canonical surjection.
- » β is the map induced by $[p^\nu]$, and is an isomorphism by the choice of μ_0.
- » γ is the canonical inclusion.

Observe that both objects in the top row are isomorphic to $\Gamma_{\nu+1}$. We conclude that the kernel of $[p^\nu]\colon \Gamma_{\nu+1} \to \Gamma_{\nu+1}$ is isomorphic to the kernel of α, which is $H_{\nu+\mu_0}/H_{\mu_0}$. By definition this is Γ_ν. We conclude that Γ is indeed p-divisible. $\square$

References

[1] J. T. Tate. “p-divisible groups”. In: *Proc. Conf. Local Fields (Driebergen, 1966)*. Springer, Berlin, 1967, pp. 158–183.