A LIQUID EXAMPLE COMPUTATION

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0.1. These examples were sketched to me by Peter Scholze. All ideas are due to him; all errors are mine. I thank Scholze for all his patient explanations.

0.2. Below, we will be looking at three observations concerning liquid vector spaces.

(*i*) We show that

$$\begin{aligned} \varphi\colon \ell^1 &\longrightarrow \ell^2/\ell^1 \\ (x_n)_n &\longmapsto (x_n \log |x_n|)_n \end{aligned}$$

is a linear map of liquid vector spaces.

(*ii*) We use this to show that $(\mathbb{R}, \mathcal{M}_1)$ is not an analytic ring.

(*iii*) The linear map given above is weird, because of the log-factor. But it cannot get much weirder: having factors x^c, for $c > 1$, leads to genuinely non-linear maps.

0.3. Fix $p \in \mathbb{R}_{>0}$. Recall that $\ell^p = \ell^p(\mathbb{N})$ denotes the subspace of sequences

$$\Big\{ (a_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \;\Big|\; \sum_n |a_n|^p < \infty \Big\}.$$

In particular, ℓ^1 consists of sequences whose series is absolutely convergent, and ℓ^2 is the space of square-summable sequences.

The definition is commonly extended to $p = \infty$, so that ℓ^∞ denotes the subspace of bounded sequences

$$\Big\{ (a_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \;\Big|\; \sup_n |a_n| < \infty \Big\}.$$

Note that for $p \le p' \le \infty$, there is an inclusion $\ell^p \subseteq \ell^{p'}$.

0.4. The spaces ℓ^p are real vector spaces, naturally equipped with a norm. For $p < \infty$, this norm is given by

$$\|(a_n)_n\|_p = \left(\sum_n |a_n|^p \right)^{1/p}.$$

For $p = \infty$, we take the sup-norm. This norm makes ℓ^p into a Banach space when $p \ge 1$.

Recall that Banach spaces are $\mathcal{M}$-complete condensed $\mathbb{R}$-vector spaces [1, Lect. III], and hence p'-liquid real vector spaces [1, Lect. VI], for any $0 < p' \leq 1$. As such, ℓ^p is p'-liquid. We will omit p' below and simply speak of liquid vector spaces.

0.5. One of the central protagonists in this example is the following function:

$$\begin{aligned} \tilde{\varphi}\colon \ell^1 &\longrightarrow \ell^2 \\ (x_n)_n &\longmapsto (x_n \log |x_n|)_n \end{aligned}$$

(If, as the author, one is not well-versed in basic calculus, then it is a good exercise to check that this function indeed lands in ℓ^2.)

Clearly, this is not a linear map. But its composition with the natural projection $\ell^2 \to \ell^2/\ell^1$ is linear! We will check this below.

0.6. Note that ℓ^2/ℓ^1 is a perfectly fine liquid vector space, albeit non-separated. It is crucial to remember that this quotient is taken in the condensed sense. In other words, ℓ^2/ℓ^1 is the sheaf $S \mapsto \ell^2(S)/\ell^1(S)$, where $\ell^p(S)$ is the group of continuous maps $S \to \ell^p$, for any profinite set S.

(It turns out that sheafification is not necessary, since ℓ^1 is a Banach space. Indeed, [1, Prop. 8.19] shows $H^1(S, V) = 0$ for all profinite S and Banach spaces V.)

0.7. Let us now check the linearity of the map φ, claimed above. Fix $x, y \in \ell^1$, and recall that

$$(\tilde{\varphi}(x+y) - \tilde{\varphi}(x) - \tilde{\varphi}(y))_n = (x_n + y_n)\log|x_n + y_n| - x_n \log|x_n| - y_n \log|y_n|.$$

In general, for $a, b \in \mathbb{R}$ we have

$$|(a+b)\log|a+b| - a\log|a| - b\log|b|| \leq 2\log(2)(|a| + |b|),$$

see [1, Lem. 5.3]. Hence

$$\sum_n (\tilde{\varphi}(x+y) - \tilde{\varphi}(x) - \tilde{\varphi}(y))_n \leq \sum_n 2\log(2)(|x_n| + |y_n|),$$

which shows that $\tilde{\varphi}(x+y) - \tilde{\varphi}(x) - \tilde{\varphi}(y) \in \ell^1$, and therefore φ is additive. Similarly, one may check φ preserves scalar multiplication. This shows that indeed

$$\begin{aligned} \varphi\colon \ell^1 &\longrightarrow \ell^2/\ell^1 \\ (x_n)_n &\longmapsto (x_n \log |x_n|)_n \end{aligned}$$

is a linear map.

0.8. For sake of completeness, observe that φ is not identically 0 either. For example, take $x \in \ell^1$ given by

$$x_n = \frac{-1}{n \log |n|^{3/2}} \quad \text{for } n \geq 1, \qquad x_0 = 0,$$

so that

$$\begin{aligned} x_n \log |x_n| &= \frac{\log |n \log |n|^{3/2}|}{n \log |n|^{3/2}} \\ &= \frac{1}{n \log |n|^{1/2}} + \frac{\log |\log |n||}{n \log |n|^{3/2}} \\ &\geq \frac{1}{n \log |n|^{1/2}}. \end{aligned}$$

Since the series $\sum_n \frac{1}{n \log |n|^{1/2}}$ diverges, we see that x is not in $\ker \varphi$.

0.9. For the computation below, we will denote by $e_i \in \ell^p$ the i-th standard basis vector, defined by

$$(e_i)_n = \delta_{n,i} = \begin{cases} 1 & \text{if } n = i \\ 0 & \text{if } n \neq i. \end{cases}$$

0.10. Let S denote the profinite set $\mathbb{N} \cup \{\infty\}$. Every converging sequence $(x_n)_n$ in some $\mathcal{M}$-complete vector space V induces a map $S \to V$, and hence a map $\mathcal{M}_1(S) \to V$. See [1, Lect. III] for details.

0.11. Consider any null sequence $(\alpha_n)_n$ of real numbers. This induces a null sequence in ℓ^1, namely $(\alpha_n e_n)_n$. Under the map φ studied above, this null sequence is mapped identically to $0 \in \ell^2/\ell^1$.

Now consider the map $f\colon \mathcal{M}_1(S) \to \ell^1$ induced by the null sequence $(\alpha_n e_n)_n$. Additionally, consider the image of $\varphi \circ f$, which is a subspace of ℓ^2/ℓ^1.

0.12. We will now show that $(\mathbb{R}, \mathcal{M}_1)$ is not an analytic ring. If it were so, then the image of $\varphi \circ f$ would have to be identically 0, since $\varphi \circ f$ would then be the canonical map $\mathcal{M}_1(S) \to \ell^2/\ell^1$ induced by the zero sequence $S \to \ell^2/\ell^1$.

But this is not what happens. Indeed, pick any $y = (y_n)_n \in \mathcal{M}_1(S)$ with $\sum_n |y_n| < \infty$. Then $f(y) = (\alpha_n y_n)_n$, and

$$\varphi(f(y)) = \alpha_n y_n \log |\alpha_n y_n|.$$

By picking

$$\alpha_n = \frac{-1}{\log(n)^{1/4}}, \qquad y_n = \frac{1}{n \log(n)^{5/4}}$$

we see that $f(y)$ is exactly the example described in 0.8, showing $\varphi(f(y)) \neq 0$.

Conclusion: $(\mathbb{R}, \mathcal{M}_1)$ is not an analytic ring.

0.13. We will now perform the same computation, but with the analytic ring $(\mathbb{R}, \mathcal{M}_{<1})$, and see what it teaches us.

Therefore, let $\tilde{\psi}\colon \mathbb{R} \to \mathbb{R}$ be a function such that

$$\begin{aligned}\psi\colon \ell^1 &\longrightarrow \ell^\infty/\ell^1\\ (x_n)_n &\longmapsto (\tilde{\psi}(x_n))_n\end{aligned}$$

is a linear map of liquid vector spaces. And as before, let $(\alpha_n)_n$ be a null sequence of real numbers, which induces a map $f\colon \mathcal{M}_{<1}(S) \to \ell^1$.

0.14. Clearly ψ vanishes on $\alpha_n e_n$. Since $(\mathbb{R}, \mathcal{M}_{<1})$ is an analytic ring, we see that the image of $\psi \circ f$ is trivial.

Let $y = (y_n)_n \in \mathcal{M}_{<1}(S)$ be any p-summable sequence, for some $p < 1$. In other words, $\sum_n |y_n|^p < \infty$. Since $\psi(f(y)) = 0$, we find that

$$(\tilde{\psi}(f(y)_n)_n = (\tilde{\psi}(\alpha_n y_n))_n$$

is in ℓ^1.

If $\tilde{\psi}$ grows asymptotically faster than x^c, for some $c' > 1$, than we can find $(\alpha_n)_n$, $p < 1$ and $(y_n)_n \in \mathcal{M}_p(S)$ such that $\sum_n \tilde{\psi}(\alpha_n y_n)$ diverges. We conclude that $\tilde{\psi}$ must be $O(x^c)$ for all $c > 1$.

0.15. Observe that in this current situation, the explicit values of α_n and y_n that were picked in 0.12 are ruled out. The sequence $y_n = (n \log(n)^{5/4})^{-1}$ is not p-summable for any $p < 1$.

0.16. Peter Scholze further pointed out that all the computations above can also be done for the pre-analytic rings $(\mathbb{Z}[T^{-1}], \mathcal{M}(\_, \mathbb{Z}((T))_r))$ and $(\mathbb{Z}[T^{-1}], \mathcal{M}(\_, \mathbb{Z}((T))_{>r}))$. The latter is analytic, but the former is not. In this context the role of $x \mapsto x \log |x|$ is played by the derivative $\sum_n a_n T^n \mapsto \sum n a_n T^{n-1}$.

References

[1] P. Scholze. "Lectures on Analytic Geometry". 2020.