Finite group schemes

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1 References

- The main reference is §3 of the manuscript of Moonen and his coauthors.
- For some useful facts on connected (resp. reduced) schemes, see EGA IV.
- If you are hardcore, the most general version of any statement about group schemes can be found in SGA3.
2 Examples

2.1 Examples we have seen before

Let $S$ be a scheme. We recall some examples of group schemes you have already seen.

- The group scheme $\mathbb{G}_a_S$ is defined by the functor

$$\mathbb{G}_a_S: \text{Sch}^{\text{op}}_{/S} \to \text{Grp}$$

$$T \mapsto (\mathcal{O}_T(T), +)$$

It is represented by the scheme $\mathbb{A}^1_S = \text{Spec}(\mathbb{Z}[X]) \times S$. If $S$ is affine, say $\text{Spec}(A)$, then $\mathbb{G}_a_S \cong \text{Spec}(A[X])$.

- The group scheme $\mathbb{G}_m_S$ is defined by the functor

$$\mathbb{G}_m_S: \text{Sch}^{\text{op}}_{/S} \to \text{Grp}$$

$$T \mapsto \mathcal{O}_T(T)^*$$

It is represented by the scheme $\mathbb{G}_m_S = \mathbb{Z} \times S = \text{Spec}(\mathbb{Z}[X, X^{-1}]) \times S$. If $S$ is affine, say $\text{Spec}(A)$, then $\mathbb{G}_m_S \cong \text{Spec}(A[X, X^{-1}])$.

If $S' \to S$ is a morphism of schemes, then $\mathbb{G}_a_S' \cong \mathbb{G}_a_S \times_S S'$ and $\mathbb{G}_m_S' \cong \mathbb{G}_m_S \times_S S'$. This is immediate from the way we gave the representing schemes in the above examples.

These examples naturally lead to the definition of the following subgroup schemes.

- The subgroup scheme $\mu_{n,S} \subset \mathbb{G}_m_S$ is defined by the functor

$$\mu_{n,S}: \text{Sch}^{\text{op}}_{/S} \to \text{Grp}$$

$$T \mapsto \{x \in \mathcal{O}_T(T)^* \mid x^n = 1\}$$

It is represented by $\text{Spec}(\mathbb{Z}[X]/(X^n - 1)) \times S$.

- Assume the characteristic of $S$ is a prime $p > 0$. (In other words, $\mathcal{O}_S(S)$ is a ring of characteristic $p$ or equivalently, $S \to \text{Spec}(\mathbb{Z})$ factors via $\text{Spec}(\mathbb{F}_p)$.)

The subgroup scheme $\alpha_{p^n,S} \subset \mathbb{G}_a_S$ is defined by the functor

$$\alpha_{p^n,S}: \text{Sch}^{\text{op}}_{/S} \to \text{Grp}$$

$$T \mapsto \{x \in \mathcal{O}_T(T)^* \mid x^{p^n} = 0\}$$

It is represented by $\text{Spec}(\mathbb{Z}[X]/(X^p)) \times S$.

In a moment we will see that $\mu_{n,S}$ and $\alpha_{p^n,S}$ are examples of kernels.

**Example 1** Observe that if we forget the group structures, then $\mu_{p^n,S}$ and $\alpha_{p^n,S}$ represent the same functor. Indeed, they are fibres of the same homomorphism of rings. However, as group schemes they are not isomorphic.
2.2 Constant group schemes

Let $G$ be an abstract group. We associate a group scheme with $G$, the so called constant group scheme $G_S$. It is defined by the functor

$$G_S: \text{Sch}_S^{\text{op}} \rightarrow \text{Grp}$$

$$T \mapsto \mathbb{G}_{\text{m}}(T)$$

It is represented by $\coprod_{g \in G} S$. Indeed, if $T$ is connected,

$$\text{Hom}_S(T, \coprod_{g \in G} S) = G_S(T)$$

because $T$ must be mapped to exactly one copy of $S$, and the mapping must be the structure morphism $T \rightarrow S$. For general $T$, the identity follows from abstract nonsense:

$$\text{Hom}(\coprod_{i \in I} T_i, X) = \prod_{i \in I} \text{Hom}(T_i, X)$$

Example 2 Let $k$ be a field of characteristic $p$. Let $n$ be an integer that is not divisible by $p$. In general $(\mathbb{Z}/n\mathbb{Z})_k$ and $\mu_{n,k}$ are not isomorphic. However, if $k$ contains a primitive $n$-th root of unity (for example if $k$ is algebraically closed), then $(\mathbb{Z}/n\mathbb{Z})_k \cong \mu_{n,k}$.

We say that $\mu_n$ is a form of the constant group scheme $(\mathbb{Z}/n\mathbb{Z})_k$. Later on we hope to see that, if $k$ is a field of characteristic $0$, then every finite group scheme over $k$ is a form of a constant group scheme. Moreover, if $k$ is algebraically closed, then every finite group scheme is constant.

2.3 Kernel of a homomorphism of group schemes

Let $f: G \rightarrow H$ be a homomorphism of group schemes over some scheme $S$. The kernel subgroup scheme $\text{Ker}(f) \subset G$ is defined via the functor

$$\text{Ker}(f): \text{Sch}_S^{\text{op}} \rightarrow \text{Grp}$$

$$T \mapsto \text{Ker}(G(T) \rightarrow H(T))$$

This functor is representable, because it is a pullback

$$\begin{array}{ccc}
S & \longrightarrow & H \\
\downarrow & & \downarrow f \\
\text{Ker}(f) & \longrightarrow & G \\
\end{array}$$

Note that $\mu_n$ is the kernel

$$[n]: \mathbb{G}_{\text{m}} \rightarrow \mathbb{G}_{\text{m}}$$

$$x \mapsto x^n$$

and similarly $\alpha_{p^n}$ is the kernel of Frobenius

$$\text{Frob}_p: \mathbb{G}_{\text{a}} \rightarrow \mathbb{G}_{\text{a}}$$

$$x \mapsto x^{p^n}$$
2.4 Multiplication by \( n \)

Let \( S \) be a scheme. Let \( G/S \) be a commutative group scheme over \( S \). For every non-negative integer \( n \in \mathbb{Z}_{\geq 0} \) there is a group scheme homomorphism “multiplication by \( n \)” given by

\[
[n]: G \rightarrow G \\
x \mapsto n \cdot x
\]

(Here we use additive notation for \( G \).)

The kernel of this morphism is usually denoted \( G[n] \).

Note that we can define \( \mu_n \) as \( G_{\text{m}}[n] \).

2.5 Semidirect product of group schemes

Let \( N \) and \( Q \) be two group schemes over a basis \( S \). Let

\[
\text{Aut}(N): \text{Sch}^{\text{op}}_S \rightarrow \text{Grp} \\
T \mapsto \text{Aut}(N_T)
\]

denote the automorphism functor of \( N \). (By the way, with \( \text{Aut}(N_T) \) we mean automorphisms of \( N_T \) as group scheme!) Let \( \rho: Q \rightarrow \text{Aut}(N) \) be an action of \( Q \) on \( N \).

The semi-direct product group scheme \( N \rtimes_\rho Q \) is defined by the functor

\[
N \rtimes_\rho Q: \text{Sch}^{\text{op}}_S \rightarrow \text{Grp} \\
T \mapsto N(T) \rtimes_{\rho_T} Q(T)
\]

which is represented by \( N \times_S Q \). Recall that if \((n, q)\) and \((n', q')\) are \( T \)-valued points of \( N \rtimes_\rho Q \), then

\[
(n, q) \cdot (n', q') = (n \cdot \rho(q)(n'), q \cdot q').
\]

3 Étale schemes over fields

3.1 Étale morphisms

We now give two definition of étale morphisms; but we do not show that they are equivalent.

**Definition 1** A morphism of schemes \( X \rightarrow S \) is étale if it is flat and unramified.

Observe that

- \( X \rightarrow \text{Spec}(k) \) is always flat (trivial);
- \( X \rightarrow \text{Spec}(k) \) is unramified if it is locally of finite type and if for all \( x \in X \) the ring map \( k \rightarrow \mathcal{O}_{X,x} \) is a finite separable field extension.

**Definition 2** A morphism of schemes \( X \rightarrow S \) is formally étale if for every

- commutative ring \( A \),
- and every ideal \( I \subset A \), such that \( I^2 = 0 \),
and every commutative square

\[
\begin{array}{ccc}
\text{Spec}(A/I) & \longrightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec}(A) & \longrightarrow & S
\end{array}
\]

there exists precisely one map \( \text{Spec}(A) \to X \) such that

\[
\begin{array}{ccc}
\text{Spec}(A/I) & \longrightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec}(A) & \longrightarrow & S
\end{array}
\]

commutes.

**Proposition 1** A morphism of schemes \( X \to S \) is étale if and only if it is locally of finite presentation and formally étale.

**Example 3** In other words, a group scheme \( G/k \) over a field \( k \) is étale if for every \( k \)-algebra \( A \), and every ideal \( I \subset A \) with \( I^2 = 0 \), the map \( G(A) \to G(A/I) \) is a bijection.

We now specialise to the case \( S = \text{Spec}(k) \), with \( k \) a field. Fix a separable closure \( \bar{k} \) of \( k \).

**Theorem 1** The functor

\[ \{ \text{ét. sch. over } k \} \longrightarrow \{ \text{disc. ctu. Gal(\bar{k}/k)-sets} \} \]

\[ X \longmapsto X(\bar{k}) \]

is an equivalence of categories.

**Proof** Every discrete \( \text{Gal}(\bar{k}/k) \)-set is a disjoint union of orbits. Every orbit is stabilised by a finite index subgroup \( H \subset \text{Gal}(\bar{k}/k) \). The orbit corresponds to \( \text{Spec}(\bar{k}^H) \).

Conversely, every étale scheme over \( k \) is the disjoint union of its connected components; and every connected étale scheme over \( k \) is a field extension.

### 3.2 Étale group schemes over fields

The theorem allows us to describe étale group schemes over \( k \) as group objects in the category of discrete \( \text{Gal}(\bar{k}/k) \)-sets. In other words, a étale group scheme \( G/k \) is fully described by

- the group \( G(\bar{k}) \), together with
- the action of \( \text{Gal}(\bar{k}/k) \) on \( G(\bar{k}) \).

Vice versa, every group discrete \( G \) together with a continuous action of \( \text{Gal}(\bar{k}/k) \) acting via automorphisms of \( G \) (or equivalently, such that the multiplication \( G \times G \to G \) is Galois equivariant) determines a étale group scheme over \( k \).
4 Standard constructions

Let \( G \) be a finite (hence affine) \( k \)-group scheme. By the rank of \( G \) we mean the \( k \)-dimension of its affine algebra \( \mathcal{O}_G(G) \). For example, \( \mu_{p,k}, \alpha_{p,k} \) and \( (\mathbb{Z}/p\mathbb{Z})_k \) all have rank \( p \).

4.1 Connected component of the identity

Let \( G/k \) be a group scheme over some field \( k \). Let \( G^0 \) denote the connected component of \( G \) that contains \( e \). One expects that \( G^0 \) is a subgroup scheme of \( G \). This is indeed true. One needs to prove that the image of \( G^0 \times_k G^0 \subseteq G \times_k G \) under the multiplication map \( m: G \times_k G \to G \) is contained in \( G^0 \).

We are done if \( G^0 \times_k G^0 \) is connected.

In general, if \( X \to S \) and \( Y \to S \) are \( S \)-schemes, and \( X \) and \( Y \) are connected, then \( X \times_S Y \) need not be connected. For example take \( \mathbb{C}/\mathbb{R} \) for \( X/S \) and \( Y/S \).

However, we have a rational point \( e \in G^0(k) \) at our disposal.

Lemma 1 Let \( X/k \) be a \( k \)-scheme that is locally of finite type. Assume \( X \) is connected and has a rational point \( x \in X(k) \). Then \( X \) is geometrically connected.

Proof Let \( L/k \) be a field extension. It suffices to show that the projection \( p: X_L \to X \) is open and closed. The properties of being open and closed are local on the target. In other words, if \( (U_i)_{i \in I} \) is an affine cover of \( X \), then \( (p^{-1}(U_i))_{i \in I} \) covers \( X_L \), and if every \( p^{-1}(U_i) \to U_i \) is open and closed, then so is \( p \).

Hence we may assume that \( X \) is affine and of finite type. Let \( Z \subseteq X \) be closed. Then there exists a field \( K \), with \( k \subseteq K \subseteq L \), and \( K/k \) finite, such that \( Z \) is defined over \( K \). Concretely, there exists a \( Z' \subseteq X_K \), such that \( (Z')_L = Z \).

Thus, for every closed (and therefore, for every open) subset of \( X \) we have reduced the question to whether \( X_K \to X \) is open and closed for finite extension \( K/k \). But \( K/k \) is finite and flat, hence so is \( X_K \to X \). But finite flat morphisms are open and closed (use HAG, Chap. III, Ex. 9.1 or EGA IV, Thm. 2.4.6.).

The lemma shows that \( G^0 \) is geometrically connected. This implies that \( (G^0)_K = (G_K)^0 \) for every field extension \( K/k \).

Moreover, \( G^0 \times_k G^0 \) is connected, by \url{http://stacks.math.columbia.edu/tag/0385}. It follows that \( G^0 \) carries a subgroup scheme structure.

Together, we have proved parts of the following theorem.

Theorem 2 (Parts of proposition 3.17 from the manuscript) Let \( G \) be a group scheme, locally of finite type over a field \( k \).

(i) The identity component \( G^0 \) is an open and closed subgroup scheme of \( G \) that is geometrically irreducible. In particular: for any field extension \( k \subset K \), we have \( (G^0)_K = (G_K)^0 \).

(ii) The following properties are equivalent:

(a1) \( G \times_k K \) is reduced for some perfect field \( K \) containing \( k \);
(a2) the ring \( \mathcal{O}_{G,e} \otimes_k K \) is reduced for some perfect field \( K \) containing \( k \);
(b1) \( G \) is smooth over \( k \);
(b2) \( G^0 \) is smooth over \( k \);
(b3) G is smooth over k at the origin.

Proof The lemma gives us most of (i). The flavour for most of (ii) can be grabbed from http://stacks.math.columbia.edu/tag/04QM. Indeed (a1) ⇒ (a2) and (b1) ⇒ (b2) ⇒ (b3) are trivial.

Example 4 (i) Let k be a non-perfect field. Let α ∈ k be an element that is not a p-th power. Observe that $G = \text{Spec}(k[X,Y]/(X^p + αY^p))$ is a closed subgroup scheme of $A^2_k$. It is reduced, but not geometrically reduced, hence not smooth. (ii) Consider $\mu_n, Q$, for $n > 2$. The connected component of the identity is geometrically irreducible (as the theorem says) but all other components split into more components after extending to $\overline{Q}$.

4.2 Component scheme

Let k be a field. Let $X/k$ be a scheme, locally of finite type.

The inclusion functor

$\{\text{ét k-schemes}\} \rightarrow \{\text{loc. fin. type Sch}_k\}$

admits a left adjoint

$\varpi_0: \{\text{loc. fin. type Sch}_k\} \rightarrow \{\text{ét k-schemes}\}$

In other words, every morphism $X \rightarrow Y$ of k-schemes, with $Y/k$ étale, factors uniquely via $X \rightarrow \varpi_0(X)$.

To understand what $\varpi_0(X)$ is, we use our description of étale k-schemes.

Fix a separable closure $\overline{k}/k$. Observe that $\text{Gal}(\overline{k}/k)$ acts on $\text{Spec}(k)$, hence on, $X_{\overline{k}} = X \times_k \text{Spec}(k)$, hence on the topological space underlying $X_k$, hence on $\pi_0(X_{\overline{k}})$.

The claim is then, that this action is continuous. Indeed, every connected component $C \in \pi_0(X_{\overline{k}})$ is defined over some finite extension $k' \subset \overline{k}$ of $k$, and therefore the stabiliser of $C$ contains the open subgroup $\text{Gal}(\overline{k}/k')$. (See the manuscript §3.27 for details.) The étale k-scheme associated with this action is $\varpi_0(X)$.

This shows that $\varpi_0$ is a functor, as claimed. It is the identity on étale k-schemes. Consequently, every map $X \rightarrow Y$ to an étale scheme induces a map $\varpi_0(X) \rightarrow Y$.

There is an obvious map $X_k \rightarrow \varpi_0(X_k)$. This map is $\text{Gal}(\overline{k}/k)$-equivariant, and therefore we get a map $X \rightarrow \varpi_0(X)$. The fibers of this map are precisely the connected components of $X$ (as open subschemes of $X$).

4.2.1 Component group

Let $G/k$ be a group scheme, locally of finite type. Since $G^0 \subset G$ is a normal subgroup scheme, there is a natural group scheme structure on $\varpi_0(G)$. In particular we get the following short exact sequence of group schemes.

$1 \rightarrow G^0 \rightarrow G \rightarrow \varpi_0(G) \rightarrow 1$
4.3 Reduced group scheme

Let $k$ be a field. Let $G/k$ be a group scheme. Let $G_{\text{red}}$ be the underlying reduced scheme of $G$.

It is natural to ask if $G_{\text{red}}$ is carries a natural group scheme structure over $k$. In general the answer is no.

However, if we assume $k$ is perfect, the answer is yes. Since $G_{\text{red}}$ is reduced, it is smooth (the theorem on connected components), and therefore geometrically reduced (again the theorem). By EGA IV 4.6.1, this implies that $G_{\text{red}} \times_k G_{\text{red}}$ is reduced, and therefore is mapped to $G_{\text{red}}$ under the multiplication map $G \times_k G \to G$.

In general $G_{\text{red}} \subset G$ is not normal! See exercise 3.2 from the manuscript. For more information about (possibly) surprising behaviour, one can take a look at http://mathoverflow.net/questions/38891/is-there-a-connected-k-group-scheme-g-such-that-g-red-is-not-a-subgroup and the following example by Laurent Moret-Bailly:

Over a field of characteristic $p > 0$, take for $G$ the semidirect product $\alpha_p \rtimes \mathbb{G}_m$ where $\mathbb{G}_m$ acts on $\alpha_p$ by scaling. Then $G$ is connected but $G_{\text{red}} = \{0\} \times \mathbb{G}_m$ is not normal in $G$.

Example copied from: http://mathoverflow.net/questions/161604/is-g-operatornamered-normal-in-g?rq=1

5 Characteristic 0 group schemes are smooth

Let $k$ be a field of characteristic 0. Let $G/k$ be a group scheme that is locally of finite type.

**Theorem 3** $G$ is reduced, hence $G/k$ is smooth.

**Proof** See Theorem 3.20 of the manuscript for a proof.

This result has some nice consequences.

- If $G/k$ is finite, then it is étale.
- If $G/k$ if finite, and $k$ is algebraically closed, $G/k$ is a constant group scheme.
- If $G/k$ is finite, then it is a form of a constant group scheme.

6 Cartier duality for finite commutative group schemes

We only present Cartier duality over fields. For a more general picture, see the manuscript §3.21 and further.

Let $k$ be a field. Let $G/k$ be a finite commutative group scheme. To $G$ we can attach the functor

$$G^D: \text{Sch}_{/S}^{\text{op}} \to \text{Grp}$$

$$T \mapsto \text{Hom}_{\text{Grp}_{/S}}(G_T, \mathbb{G}_m_T)$$
If \( G \) is commutative, finite, then \( G^D \) is representable.

To see this, first remark that since \( G \) is finite over \( k \), \( G \) is affine. We can thus study \( G \), by studying its Hopf algebra.

### 6.1 Hopf algebras

I am not going to discuss Hopf algebras in the generality that mathematical physicists would do.

The category of affine \( k \)-schemes is dual to the category of \( k \)-algebras. Hence a group object in the former corresponds to a cogroup object in the latter.

In particular, for an algebra \( A \) we get the following data

- **unit (algebra structure map)** \( e : k \to A \)
- **multiplication** \( m : A \otimes_k A \to A \)

and if \( A \) is a Hopf algebra, we moreover have

- **co-unit (augmentation map)** \( \tilde{e} : A \to k \)
- **co-multiplication** \( \tilde{m} : A \to A \otimes_k A \)
- **co-inverse** \( \tilde{i} : A \to A \)

I am not going to spell out what it means for \( A \) to be a co-commutative Hopf algebra, but you will just have to dualize all diagrams for group objects.

On \( k \)-algebras, use \( (\mathcal{D}) \) as notation for the dualisation functor \( \text{Hom}(\_ , k) \).

**Lemma 2** Let \( A \) be a co-commutative Hopf algebra over \( k \). The dual data \((A^D, e^D, \tilde{m}^D, e^D, m^D, \tilde{i}^D)\) specifies a co-commutative \( k \)-Hopf algebra.

**Proof** Draw all the diagrams for a co-commutative Hopf algebra. Reverse all the arrows. Remark that nothing happened, up to a permutation.

We return to the group scheme \( G/k \). Recall that it is commutative and finite. Hence the global sections \( O_G(G) \) form a co-commutative Hopf algebra.

**Theorem 4** The Cartier dual \( G^D \) is represented by \( \text{Spec}(A^D) \).

**Proof** Let \( R \) be any \( k \)-algebra. We have to show that \( G^D(R) \) is naturally isomorphic to \( \text{Hom}_k(\text{Spec}(R), \text{Spec}(A^D)) \).

Observe that

\[
G^D(R) = \text{Hom}_{\text{GrpSch}/k}(G_R, \mathbb{G}_m) \subset \text{Hom}_R(R[x, x^{-1}], A \otimes_k R).
\]

On the other hand,

\[
\text{Hom}_k(\text{Spec}(R), \text{Spec}(A^D)) \cong \text{Hom}_k(A^D, R) \\
\cong \text{Hom}_R(A^D \otimes_k R, R) \\
\cong \text{Hom}_R(A \otimes_k R^D, R).
\]

To make life easier, we now just write \( A \) for the \( R \)-Hopf algebra \( A \otimes_k R \). So we want to prove that \( \text{Hom}_R(A^D, R) \) is canonically isomorphic to the subset of Hopf algebra homomorphisms of \( \text{Hom}_R(R[x, x^{-1}], A) \).

This latter subset is described as follows: A ring homomorphism \( f \) is determined by the image of \( x \). It is a Hopf algebra homomorphism, precisely when \( \tilde{m}(f(x)) = f(x) \otimes f(x) \).
So we get the set \( \{ a \in A^* \mid \tilde{m}(a) = a \otimes a \} \). From the diagrams for Hopf algebras, we see that if \( a \in A \) satisfies \( \tilde{m}(a) = a \otimes a \), then \( \tilde{e}(a) \cdot a = a \), and \( \tilde{i}(a) \cdot a = \tilde{e}(a) \). If \( a \in A^* \), then \( \tilde{e}(a) = 1 \) by the first equation. If on the other hand \( \tilde{i}(a) = 1 \), then the second equation implies \( a \in A^* \). Therefore

\[
\{ a \in A^* \mid \tilde{m}(a) = a \otimes a \} = \{ a \in A \mid \tilde{m}(a) = a \otimes a \text{ and } \tilde{e}(a) = 1 \}.
\]

Reasoning from the other side, every \( R \)-module homomorphism \( A^D \rightarrow R \) is an evaluation homomorphism

\[
ev_a : A^D \rightarrow R \quad \lambda \mapsto \lambda(a)
\]

If we want \( ev_a \) to be a ring homomorphism, it should satisfy

\[
ev_a(1) = 1, \quad \text{i.e. } ev_a \circ \tilde{e}^D = \text{id}
\]

\[
ev_a(\lambda \mu) = ev_a(\lambda ev_a(\mu)), \quad \text{i.e. } ev_a \circ \tilde{m}^D = ev_a \cdot ev_a.
\]

The first equation is equivalent with \( \tilde{e}(a) = 1 \). Indeed

\[
ev_a(\tilde{e}^D(1_R)) = \tilde{e}(a).
\]

The second equation demands

\[
(\lambda \otimes \mu)(\tilde{m}(a)) = \lambda(a)\mu(a)
\]

This is equivalent with \( \tilde{m}(a) = a \otimes a \), which can be seen by letting \( \lambda \) and \( \mu \) run through a dual basis of \( A \).

Hence we are back at the set

\[
\{ a \in A \mid \tilde{m}(a) = a \otimes a \text{ and } \tilde{e}(a) = 1 \}
\]

which completes the proof.

7 Exercises

(1) Show that \( \mu_n/k \) is unramified if \( n \in k^* \).

(2) Show that \( \mu_n/k \) is formally étale if \( n \in k^* \).

(3) Let \( k \) be a field of characteristic \( p \). Give a \( k \)-algebra \( A \), such that \( \alpha_p(A) \) is not trivial.

(4) Compute \( G^0 \) for \( G = \mu_n/k \) (think about the characteristic of \( k \)).

(5) Compute the Cartier dual of \( (\mathbb{Z}/n\mathbb{Z})_k \) for \( n \in k^* \).