Condensed mathematics — Seminar ws19/20

Condensed mathematics is the latest and greatest that has been flowing out of Bonn recently. I propose to organise a seminar in which we may seek to understand what it is all about.

A first motivation for condensed mathematics is the observation that the category of topological abelian groups is not well-behaved: let A be a non-trivial abelian group, and consider it as topological group endowed with the discrete topology (notation: $A_\perp$) or the trivial topology (notation: $A_\top$). Then the identity map $A_\top \to A_\perp$ is a continuous homomorphism that is injective (hence a monomorphism) and surjective (hence an epimorphism) but it is not an isomorphism in the category of topological abelian groups. In other words, this category is not an abelian category, and kernels and cokernels do not behave as we would wish.

Dustin Clausen proposed a solution to this problem, and together with Peter Scholze he has been working out the details. The result has been given the name "condensed mathematics". Peter Scholze gave a lecture course on this topic in the summer semester of 2019.

Lecture notes available at: `https://www.math.uni-bonn.de/people/scholze/Condensed.pdf`

Surprisingly the theory does not only lead to a satisfactory abelian category that contains the category of topological abelian groups as a subcategory. There is more! For example, the theory is compatible with Pontrjagin duality, and there are some surprises when computing condensed cohomology. But there is even more! Peter Scholze also develops a new proof of finiteness of coherent cohomology and Grothendieck duality. The resulting proof follows almost effortlessly from the theory developped over a few lectures.

In this seminar we will roughly follow the aforementioned lecture notes, but we will allow ourselves to digress into prerequisites when this is deemed beneficial.

1. Condensed sets: motivation and overview

(*i*) The problem; (*ii*) the proposed solution.

(*iii*) Recall (briefly!) the history of the pro-étale topology. (We don't need this for the rest of the seminar.) (*iv*) Give an explicit description of condensed sets. (In other words, what does it mean *concretely* for a presheaf to satisfy the sheaf condition on $*_{\text{proét}}$?) (*v*) If time permits, briefly recall the notion of site and topos.

(*vi*) Show how to attach a condensed set/group/ring to a topological space/group/ring. (*vii*) Observe that the resulting functor is faithful, and full when restricted to compactly generated spaces.

(*viii*) Recall Grothendieck's axioms for abelian-like categories (Tohoku math). (*ix*) Conclude that condensed abelian groups form a nice category: it satisfies (AB3), (AB3*), (AB4), and (AB5).

As usual, there are set-theoretic issues. I suggest that we don't emphasise these (or their solutions), but instead focus on the general machinery of sheaves/topoi, and the category of abelian sheaves.

2. Condensed abelian groups

(*i*) Show that the category of condensed abelian groups also satisfies (AB4*) and (AB6). (*ii*) Explicitly mention the theorems that profinite sets, compact Hausdorff spaces, and extremely disconnected spaces all give rise to the same sheaf topos, namely condensed sets. (As before, there are nasty set-theoretic issues. Try to distill the idea of the proof, and present this to the audience.) (*iii*) Compare with the *pyknotic* objects of Barwick and Haine: `https://arxiv.org/pdf/1904.09966.pdf`

(*iv*) Discuss extra structure on Cond(Ab): the tensor product, and internal Hom. (*v*) Show that we can construct the derived category $D(\mathrm{Cond}(\mathrm{Ab}))$. Scholze warns that probably $D(\mathrm{Cond}(\mathrm{Ab})) \ncong \mathrm{Cond}(D(\mathrm{Ab}))$. The solution is to use ∞-categories.

(*vi*) Give a general overview of ∞-categories; how to use them; and the problems that they solve.

Given the background of our audience, it might be edifying to spend ample time on this last point.

3. Cohomology

In the next talk we want to compare locally compact abelian groups with condensed abelian groups. In particular, we will see that there is a fully faithful functor $D^b(\mathrm{LCA}) \to D(\mathrm{Cond}(\mathrm{Ab}))$. Part of the ingredients in this comparison is an understanding of condensed cohomology.

(*i*) Recall the differences and comparisons between sheaf/Čech/singular cohomology for "reasonable space" (CW complexes, profinite sets).

Let S be a compact Hausdorff space. (*ii*) Define condensed cohomology; (*iii*) show that condensed cohomology with coefficients in $\mathbb{Z}$ coincides with sheaf cohomology; and (*iv*) show that for real coefficients the higher cohomology vanishes.

4. Locally compact abelian groups

(*i*) Recall the structure theorem for LCA: If $A \in \mathrm{LCA}$, then there exists a short exact sequence $0 \to \mathbb{R}^n \oplus C \to A \to D \to 0$, where C is compact and D is discrete. (*ii*) Recall the statement of Pontrjagin duality. (*iii*) Show that the compact-open topology is compatible with the internal Hom in Cond(Ab). (*iv*) Conclude that the functor $\mathrm{LCA} \to \mathrm{Cond}(\mathrm{Ab})$ preserves Pontrjagin duality. (*v*) Explain how to compute $R\underline{\mathrm{Hom}}(A, B)$ for locally compact abelian groups A and B. (*vi*) Say something about how to construct $D^b(\mathrm{LCA})$. [Warning: LCA is not abelian; so this is non-trivial!] (*vii*) Construct the functor $D^b(\mathrm{LCA}) \to D(\mathrm{Cond}(\mathrm{Ab}))$ and show that it is fully faithful.

5. Solid abelian groups

We now come to a part of the seminar where it is not immediately clear how to apply intuition from "old-school maths". The best intuition that I have so far is that "solidification" should be analogue of compactification.

This talk will probably benefit from coordination with the next talk's speaker.

(*i*) Introduce *solid* abelian groups. (*ii*) Define the free solid abelian group on a profinite set, discuss its structure, and prove that it is solid. (*iii*) State theorem 5.8 of the lecture notes, which describes categorical properties of the category Solid; and (*iv*) sketch a proof of the theorem.

6. Solid abelian group continued

This talk will probably benefit from coordination with the previous talk's speaker.

(i) Show that the subcategory of compact objects in $D(\text{Solid})$ is anti-equivalent to $D(\mathbb{Z})$. (ii) Define the completed tensor product, and show that (derived) solidification is monoidal. (iii) Let $\mathbb{Q}_v$ and $\mathbb{Q}_{v'}$ be two completions of $\mathbb{Q}$. Compute $\mathbb{Q}_v \otimes^{L\blacksquare} \mathbb{Q}_{v'}$. ($iv$) Show that the singular homology complex of a CW complex X is isomorphic to the derived solidification complex of the free abelian group on X.

7. Analytic rings

(i) Define pre-analyic rings; (ii) try to motivate the definition; (iii) give examples.

(iv) Define analytic rings. (v) Describe the (derived) category of condensed modules over an analytic ring. (vi) Discuss examples.

Analytic rings will play an important role in the last part of the seminar. It might be good to spend more time on motivating the definition and explaining a point of view, rather than getting lost in the proof of some categorical property.

In particular, it will be helpful to discuss geometric intuition for $A_\blacksquare$ and $(A, \mathbb{Z})_\blacksquare$; how they relate, and how they differ. They are protagonists in the remainder of the seminar.

8. Solid modules

We are now getting to the part of the seminar that revolves around coherent duality, and the new approach to it that condensed mathematics offers.

(i) State the theorems that describe j_*, j^*, $j_!$, and $f_!$, $f^!$. (ii) Point out that this is a surprise. In the classical setup the functor $f_!$ does not even exist.

(iii) Try to prove as much as possible in the special case $A = \mathbb{Z}[T]$. See the observations in §8 of the lecture notes by Scholze. (It is more important to explain structure and intuition, then to give detailed proofs.) (iv) If time permits, contrast the proof of the special case with that of the general case.

9. Globalization

(i) Give a recap of Huber pairs. (ii) Define the functors $X \mapsto X^{\text{ad}}$ and $X \mapsto X^{\text{ad}/R}$ and describe their functors of points.

Let X be a discrete adic space. (iii) Comment on the category $\mathcal{D}((\mathcal{O}_X, \mathcal{O}_X^+)_\blacksquare)$. ($iv$) Explain the obstruction to gluing sheaves of modules, or even derived categories. (v) Motivate the ∞-categorical fix.

10. Globalization continued

(i) Show that the ∞-categorical derived categories of modules form a sheaf. (ii) Define the category $\mathcal{D}((\mathcal{O}_X, \mathcal{O}_X^+)_\blacksquare)$. ($iii$) Give a concrete description of $D((\mathcal{O}_X, \mathcal{O}_X^+)_\blacksquare)$.

11. Coherent duality

(*i*) Give the statement of coherent duality and recall how it generalises Serre duality. (*ii*) Recall the 6-functor formalism. (*iii*) Explain how to define the functors $f_!$ and $f^!$ using solid modules. (*iv*) Sketch the proof that for smooth $f\colon X \to Y$ of dimension d we have $f^!\mathcal{O}_Y = \omega_{X/Y}[d]$.

The End

Thanks for reading this far! I am looking forward to a stimulating seminar.

Johan Commelin Freiburg, Summer 2019