MACLANE'S Q'-CONSTRUCTION AND BREEN–DELIGNE RESOLUTIONS (DRAFT)

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1. Introduction

1.1. The purpose of this note is to construct a functorial complex $Q'(A)$ with the following two properties:

(i) Every object $Q'(A)_i$ is of the form $\bigoplus_{j=1}^{n_i} \mathbb{Z}[A^r_{i,j}]$.

(ii) Modulo some details that will be made precise below, we have

$$\mathrm{RHom}(Q'(A), B) = 0 \qquad \Longrightarrow \qquad \mathrm{RHom}(A, B) = 0.$$

See Lemma 4.4 for the actual statement.

2. Breen–Deligne resolutions

2.1. **Theorem** (Breen, Deligne, [2, Appendix to §IV])**.** *There exists a functorial resolution* $\mathrm{BD}(A)$ *of an abelian group A of the form*

$$\cdots \to \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}] \to \cdots \to \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0$$

where all n_i and $r_{i,j}$ are natural numbers.

Proof. See the appendix to Lecture IV in [2]. The proof uses a nontrivial amount of homotopy theory. □

2.2. **Remark.** The map $\mathbb{Z}[A] \to A$ is simply the evaluation morphism $\sum c_a[a] \mapsto \sum c_a a$. The kernel of this map is generated by elements of the form $[a] + [b] - [a + b]$. In particular, the map $\mathbb{Z}[A^2] \to \mathbb{Z}[A]$ is the map induced by $(a, b) \mapsto [a] + [b] - [a + b]$.

The kernel of that map is generated by elements of the form

$$[a, b + c] + [b, c] - [a, b] - [a + b, c] \qquad \text{and} \qquad [a, b] - [b, a].$$

The fact that this process can be continued to form a functorial resolution is the non-trivial content of Theorem 2.1.

2.3. Breen–Deligne resolutions have some very favourable properties, which have been used in [2].

(i) First and foremost, they are functorial in the abelian group A.

(ii) They exist in the generality of abelian group objects in any sheaf topos.

(iii) There exists a functorial homotopy between the "outer" and "inner" addition maps $\mathrm{BD}(A^2) \to \mathrm{BD}(A)$. See 3.6.

2.4. Philosophical remark: Due to the inexplicit nature of the proof of Theorem 2.1, these Breen–Deligne resolutions cannot be used to compute explicit values of derived functors apart from vanishing results. Indeed, it seems that this is how Breen–Deligne resolutions are typically applied.

3. The engine

3.1. We now give a setup in which we can give an elementary alternative to Breen–Deligne resolutions. This setup applies to (condensed) abelian groups. It could be generalised further, but I do no know of elementary proofs in the general setting.

3.2. Let $\mathscr{A}$ be an abelian category with enough projectives, and assume that there is an action $\mathrm{Ab} \otimes \mathscr{A} \to \mathscr{A}$ that preserves coproducts in the first factor and such that $\mathbb{Z} \otimes -$ is naturally isomorphic to the identity. NB: this forces $\mathscr{A}$ to have arbitrary coproducts.

Finally, assume that $\mathscr{A}$ is an AB4-category, so that $\mathrm{Ext}^i(-,-)$ will turn coproducts in the first entry into products.

3.3. **Lemma.** *Let A and B be objects of $\mathscr{A}$, and let $C \in \mathrm{Ch}_{\geq 0}(\mathscr{A})$ be a chain complex. Assume that*

- *$H_0(C) \cong A$;*
- *for all $i > 0$, there exists an abelian group H such that $H_i(C) \cong H \otimes A$;*
- *the functor $- \otimes A$ is exact.*

Let j be a natural number. Then the following implication holds: If $\mathrm{Ext}^i(C, B) = 0$ for all $i \leq j$ then also $\mathrm{Ext}^i(A, B) = 0$ for all $i \leq j$.

Proof. We induct on j. For $j = 0$, note that every homomorphism $C \to B$ factors uniquely over $H_0(C)$. Since we assumed $H_0(C) \cong A$, we are done.

Now assume the result is true for j. Let $\mathscr{S}$ be the class of all complexes $T \in \mathrm{Ch}_{\geq 0}(\mathscr{A})$ for which $\mathrm{Ext}^i(T, B) = 0$ for all $i \leq j + 1$. We assume that $C \in \mathscr{S}$, and we want to conclude $A \in \mathscr{S}$.

The class $\mathscr{S}$ has the following two properties:

(i) It is closed under arbitrary coproducts (since $\mathscr{A}$ is AB4).
(ii) If $T_1 \to T_2 \to T_3 \xrightarrow{+1}$ is a triangle and $T_1 \in \mathscr{S}$ then $T_2 \in \mathscr{S} \iff T_3 \in \mathscr{S}$ (by the long exact sequence of Ext-groups).

Now consider the triangles

$$\Delta_k \colon \tau_{\geq k+1} C \to \tau_{\geq k} C \to H_k(C)[k] \xrightarrow{+1}$$

For $k > j$, we know that $\tau_{\geq k+1} C \in \mathscr{S}$.

We will be done if we show $H_k(C)[k] \in \mathscr{S}$ for $k > 0$. Indeed, if that is the case, we have $\tau_{\geq k} C \in \mathscr{S}$ for all $k > 0$, by descending induction on k and the closure property of $\mathscr{S}$ for the triangles Δ_k. Finally, since $\tau_{\geq 0} C = C$, we use Δ_0 to conclude $H_0(C)[0] \cong A \in \mathscr{S}$.

Let $k > 0$. We want to show $H_k(C)[k] \in \mathscr{S}$. By assumption, it suffices to show that $H \otimes A[k] \in \mathscr{S}$ for arbitrary abelian groups H. Furthermore, we point out that $A[k] \in \mathscr{S}$, by the induction hypothesis. Indeed, $\mathrm{Ext}^i(A[k], B) = \mathrm{Ext}^{i-k}(A, B) = 0$ for all $i \leq j + 1$ since $i - k \leq j$.

Let H be any abelian group, and consider a two-step free resolution

$$0 \to \bigoplus_{s \in S_1} \mathbb{Z} \to \bigoplus_{s \in S_0} \mathbb{Z} \to H \to 0.$$

Since $-\otimes A$ is exact and preserves coproducts, we obtain a short exact sequence

$$0 \to \bigoplus_{s\in S_1} A \to \bigoplus_{s\in S_0} A \to H\otimes A \to 0.$$

We conclude that $H \otimes A[k] \in \mathscr{S}$, since $A[k] \in \mathscr{S}$. In particular, $H_k(C)[k]$ is contained in $\mathscr{S}$, which finishes the proof. □

3.4. We now wish to find situations where Lemma 3.3 can be applied. First consider the case $\mathscr{A} = \mathrm{Ab}$. In that case, the condition that $-\otimes A$ is exact means that we should consider flat abelian groups, a.k.a. torsion-free abelian groups.

If A is torsion-free, then it is naturally a filtered colimit of finitely-generated free groups. (Indeed, it is the union of its finitely generated subgroups, which are free.)

3.5. Let $C\colon \mathrm{Ab} \to \mathrm{Ch}_{\geq 0}(\mathrm{Ab})$ be a functor. Then there is a natural map

$$\begin{aligned} A &\to \mathrm{Hom}(H_k(C(\mathbb{Z})), H_k(C(A))) \\ a &\mapsto H_k(C(1\mapsto a)), \end{aligned}$$

inducing a natural map $H_k(C(\mathbb{Z}))\otimes A \to H_k(C(A))$. Suppose that $H_k(C(-))$ is additive and preserves filtered colimits. Then this natural map is an isomorphism for torsion-free abelian groups A.

Note that H_k is additive and preserves filtered colimits. Below, we will construct and example of a functor C that preserves filtered colimits, and is *additive up to homotopy.* This is good enough, because the composition $H_k(C(-))$ will then be additive.

3.6. We say that C is *additive up to homotopy* if the following condition is satisfied. For every abelian group A, there is a natural map $\sigma\colon C(A^2) \to C(A)$ induced by the addition map $+\colon A^2 \to A$. On the other hand, there is also a natural "addition on the outside", obtained by adding the two maps $C(A^2) \to C(A)$ induced by the projection maps $\pi_1, \pi_2\colon A^2 \to A$.

Indeed, the addition map $+\colon A^2 \to A$ is the sum $\pi_1 + \pi_2$, and hence σ is equal to $C(\pi_1+\pi_2)$. The "addition on the outside" is the map $\pi \overset{\mathrm{def}}{=} C(\pi_1) + C(\pi_2)$.

If C is additive, then $\sigma = \pi$. We say that C is *additive up to homotopy* if σ and π are homotopic for all A.

4. MacLane's Q' construction

4.1. Let $\mathscr{A}$ be an abelian category and let $F\colon \mathscr{A}\to\mathscr{A}$ be a functor. We will think of $F(A)$ as the "free" object generated by A. Indeed, the typical example is $\mathscr{A} = \mathrm{Mod}_R$ and $F(M) = R[M]$.

For any $A \in \mathscr{A}$, let $\pi_1, \pi_2\colon A^2 \to A$ be the two projection maps, and define

$$\pi = F(\pi_1) + F(\pi_2), \qquad \sigma = F(\pi_1 + \pi_2).$$

Note that $\pi_1+\pi_2$ is the addition map $A^2 \to A$.

4.2. **Construction.** We define a functorial complex

$$Q'_F(A)\colon \quad \cdots \to F(A^{2^i}) \to \cdots \to F(A^4) \to F(A^2) \to F(A)$$

that is additive up to homotopy and such that the components of the homotopy between σ and π are the identity. This characterises Q'_F uniquely.

Indeed, the homotopy condition

$$\pi - \sigma = h_i \circ d_{i-1}(A^2) + d_i(A)\circ h_{i+1}$$

simplifies to $\pi - \sigma = d_{i-1}(A^2) + d_i(A)$, from which we find a recursive definition for the differentials $d_i(A)$.

4.3. Usually, the functor F is clear from the context, and we will simply write Q' for Q'_F.

The complex Q' is known as MacLane's Q'-construction, and appears already in §12 of [1].

4.4. **Lemma.** *Let A and B be abelian groups, and assume A is torsion-free. Let $F\colon \mathrm{Ab} \to \mathrm{Ab}$ be the functor $\mathbb{Z}[-]$ and consider $Q' = Q'_F$. If $\mathrm{RHom}(Q'(A), B) = 0$ then $\mathrm{RHom}(A, B) = 0$.*

Proof. We apply Lemma 3.3 to obtain the result. Let us check the conditions. Indeed, Ab is AB4, and $- \otimes A$ is exact because A is torsion-free.

Since $\mathbb{Z}[-]$ preserves filtered colimits, we see that $H_k(Q'(-))$ preserves filtered colimits. It is additive because $Q'(-)$ is additive up to homotopy. Therefore $H_k(Q'(A)) \cong H_k(Q'(\mathbb{Z})) \otimes A$, because A is torsion-free.

Finally observe that $H_0(Q'(A)) \cong A$, as A is naturally the cokernel of

$$\begin{aligned} \mathbb{Z}[A^2] &\to \mathbb{Z}[A] \\ [a, b] &\mapsto [a] + [b] - [a + b]. \end{aligned}$$ □

5. Condensed abelian groups

5.1. Suppose that $\mathscr{A} = \mathrm{Cond}(\mathrm{Ab})$. This is again an AB4-category, and there is a natural action $\mathrm{Ab} \otimes \mathrm{Cond}(\mathrm{Ab}) \to \mathrm{Cond}(\mathrm{Ab})$: for $H \in \mathrm{Ab}$ and $A \in \mathrm{Cond}(\mathrm{Ab})$ the presheaf $S \mapsto H \otimes A(S)$ is already a sheaf.

Let $F\colon \mathrm{Cond}(\mathrm{Ab}) \to \mathrm{Cond}(\mathrm{Ab})$ be the functor $\mathbb{Z}[-]$ that sends A to the sheafification of $S \mapsto \mathbb{Z}[A(S)]$. We will consider $Q' = Q'_F$ in this section.

5.2. **Lemma.** *Let A and B be condensed abelian groups, and assume $A(S)$ is torsion-free for all extremally disconnected S. If $\mathrm{RHom}(Q'(A), B) = 0$ then $\mathrm{RHom}(A, B) = 0$.*

Proof. We wish to apply Lemma 3.3 again. On the level of presheaves, we have a natural isomorphism between $S \mapsto H_k(Q'(A(S)))$ and $S \mapsto H_k(Q'(\mathbb{Z})) \otimes A(S)$. Thus the same is true after sheafification.

The other conditions are similarly easy to verify. □

6. A cubical construction of Q'

6.1. **Remark.** We will now give a different construction of Q' as the alternating face map complex of a semi-simplicial complex attached to a natural cubical complex. This remark and the lemma that follows it are not essential for the rest of the note.

Let $\square = \{0, 1\}$ denote a set with two elements. Then we can consider $\square^n$ as a discrete cube. For every n and every $0 \leq i \leq n$ and every $b \in \square$ we have natural maps $f_b^{n,i}\colon \square^n \to \square^{n+1}$ that maps $\square^n$ to the (i, b)-th face of $\square^{n+1}$. Concretely

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{i-1}, b, x_i, \dots, x_n).$$

If we have some object $A \in \mathscr{A}$, then we get natural maps $(f_b^{n,i})^*\colon A^{\square^{n+1}} \to A^{\square^n}$ by pullback (aka composition).

Abstractly, we can say that $A^{\square^\bullet}$ is a cubical object. From this cubical object, we are going to build a chain complex

$$\cdots \to \xrightarrow{d_n} F(A^{\square^n}) \to \ldots \xrightarrow{d_1} \underbrace{F(A^{\square^1})}_{=F(A^\square)} \xrightarrow{d_0} \underbrace{F(A^{\square^0})}_{=F(A)}$$

Since A is an object of an abelian category, we can consider the morphism

$$\sigma^{n,i} := (f_0^{n,i})^* + (f_1^{n,i})^* \colon A^{\square^{n+1}} \to A^{\square^n}$$

Now we define

$$d_n := \sum_i (-1)^i \cdot (F((f_0^{n,i})^*) + F((f_1^{n,i})^*) - F(\sigma^{n,i}))$$

This is the differential in the complex above.

6.2. **Lemma.** *This cubical construction yields a chain complex that is naturally isomorphic to $Q'(A)$.*

Proof. Certainly, the two complexes have the same differentials in degree 0, namely $F(\pi_1) + F(\pi_2) - F(\pi_1 + \pi_2)$. We leave it as an exercise to the reader to verify recursively that the other differentials also agree. □

References

[1] Samuel Eilenberg and Saunders MacLane. "Homology theories for multiplicative systems". English. In: *Trans. Am. Math. Soc.* 71 (1951), pp. 294–330.

[2] P. Scholze. "Lectures on Condensed Mathematics". 2019.