LIQUID TENSOR EXPERIMENT: BLUEPRINT FOR THE REDUCTION TO THEOREM 9.4

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Abstract. This is a detailed blueprint for the reduction of the Liquid Tensor Experiment to Theorem 9.4 of [Sch20] whose formalization was achieved in June 2021. The argument presented here is a simplified version of the original argument, and avoids the use of stable homotopy theory by replacing abstract Breen–Deligne resolutions with the explicit Q'-construction of MacLane.

1. Introduction

In [Sch21], the second author posed the challenge to formally verify the following theorem from [Sch20] (a special case of [Sch20, Theorem 9.1]):

Theorem 1.1. *Let $0 < p' < p \leq 1$ be real numbers, let S be a profinite set, and let V be a p-Banach space. Denoting by $\mathcal{M}_{p'}(S)$ the space of p'-measures on S regarded as a condensed real vector space, the groups*

$$\mathrm{Ext}^i_{\mathrm{Cond(Ab)}}(\mathcal{M}_{p'}(S), V) = 0$$

vanish for $i > 0$.

Thus, the goal is a computation of certain Ext-groups in the abelian category of condensed abelian groups; the latter is a variant of the category of topological abelian groups with much better categorical properties (in particular, being abelian).

The Lean/mathlib community has taken on the challenge, and in particular has fully formalized the key part of the proof, which is [Sch20, Theorem 9.4]. The purpose of this manuscript is to outline the remaining work, and in the process take the opportunity to outline some of the ideas involved. A mathematical contribution is a simplification of the part of the proof using Breen–Deligne resolutions; this was discovered during discussions between the authors about the formalization of [Sch20, Theorem 9.4]. We warn the reader that mathematically, this manuscript is anticlimactic: It ends where it actually gets to the heart of the matter – but that part had already been done.

2. Condensed Abelian Groups

The category of condensed abelian groups is the category of abelian sheaves on the site of profinite sets.[1] Most of the material of this section has been formalized in Lean by Adam Topaz; we indicate proofs only when the result is not yet formalized.

To get started, we recall the following result.

[1]There are some universe issues to be handled here. We believe that they are merely distracting, and leave those to the formalization.

Proposition 2.1. *The category of totally disconnected compact Hausdorff spaces is naturally equivalent to the* Pro*-category of finite sets, via sending a pro-finite set* "$\varprojlim_i$"S_i *to* $S = \varprojlim_i S_i$*, the inverse limit taken in the category of topological spaces.*

We denote their category by ProFin. We make ProFin into a site by declaring a cover to be a finite family of jointly surjective maps $\{f_i : S_i \to S\}$.

Definition 2.2. The category Cond(Ab) of condensed abelian groups is the category of abelian sheaves on ProFin. The category of condensed sets Cond is the category of sheaves (of sets) on ProFin.

The intuition here is that a condensed set/abelian group X is packaging the data of "continuous maps from any profinite set S into X", subject to some simple axioms. Unraveling the sheaf axioms we have the following description.

Proposition 2.3. *A condensed set (resp. abelian group) is a functor*

$$X : \mathrm{ProFin}^{\mathrm{op}} \to \mathrm{Set}\ (\mathrm{resp.\ Ab})$$

that preserves finite products (equivalently, $X(\emptyset) = *$ *and* $X(S_1 \sqcup S_2) = X(S_1) \times X(S_2)$*) and for any surjective map* $f : T \to S$*, the map* $X(S) \to X(T)$ *is injective with image all those* $x \in X(T)$ *whose two pullbacks to* $X(T \times_S T)$ *agree.*

By the Yoneda lemma (and a verification of the sheaf property in this case), any profinite set S defines a condensed set, given by the functor

$$\mathrm{ProFin}^{\mathrm{op}} \to \mathrm{Set} : S' \mapsto \mathrm{Hom}_{\mathrm{ProFin}}(S', S).$$

We will in the following tacitly identify profinite sets as a full subcategory of condensed sets. In a suitable sense, condensed sets are freely generated from profinite sets (subject to the gluing conditions on profinite sets encoded in the covering condition).

More generally, any topological space defines a condensed set:[2]

Proposition 2.4. *For any topological space* X*, the functor*

$$\underline{X} : \mathrm{ProFin}^{\mathrm{op}} \to \mathrm{Set} : S \mapsto \mathrm{Cont}(S, X)$$

is a condensed set. Similarly, a topological abelian group M *defines a condensed abelian group* $\underline{M}$*.*

The following proposition holds more generally for sheaves on any site.

Proposition 2.5. *The category of condensed abelian groups is an abelian category. The forgetful functor* Cond(Ab) $\to$ Cond *admits a left adjoint* $X \mapsto \mathbb{Z}[X]$*, given as the sheafification of* $S \mapsto \mathbb{Z}[X(S)]$*.*

An important result will be the explicit description of the free condensed abelian groups $\mathbb{Z}[S]$ on profinite sets S; this is the first result that has not been formalized.

[2]Possibly up to universe issues.

Proposition 2.6. *Let $S = \varprojlim_i S_i$ be a profinite set, written as a cofiltered inverse limit of finite sets S_i. Then the map*

$$\mathbb{Z}[S] \to \varprojlim_i \mathbb{Z}[S_i]$$

of condensed abelian groups is injective, and the image can be described as follows. For any $n \geq 0$, let $\mathbb{Z}[S_i]_{\leq n} \subset \mathbb{Z}[S_i]$ be the subset of those sums $\sum_{s \in S_i} n_s[s]$ with $\sum_{s \in S_i} |n_s| \leq n$; this is (the condensed set corresponding to) a finite set. Let

$$\mathbb{Z}[S]_{\leq n} := \varprojlim_i \mathbb{Z}[S_i]_{\leq n} \subset \varprojlim_i \mathbb{Z}[S_i]$$

Then

$$\mathbb{Z}[S] = \bigcup_n \mathbb{Z}[S]_{\leq n} \subset \varprojlim_i \mathbb{Z}[S_i].$$

Proof. See Proposition 2.1 of [Sch20]. □

A very pleasant property of condensed sets is the presence of enough projective objects.

Proposition 2.7. *A profinite set S is projective in the category of profinite sets, i.e. any surjection $f : T \to S$ splits, if and only if S is extremally disconnected, i.e. the closure of any open subset is open. Any profinite S admits a surjection from an extremally disconnected profinite set.*

Let ExtrDisc ⊂ ProFin be the subcategory of extremally disconnected profinite sets.

Proposition 2.8. *Restricting along* ExtrDisc ⊂ ProFin *presents* Cond *(resp.* Cond(Ab)*) as the category of finite-product preserving functors*

$$\mathrm{ExtrDisc}^{\mathrm{op}} \to \mathrm{Set}\ (\text{resp. Ab}).$$

Corollary 2.9. *For any $S \in$ ExtrDisc, the condensed abelian group $\mathbb{Z}[S] \in$ ExtrDisc is projective; this class of objects generates* Cond(Ab).

In particular, one can define Ext-groups in Cond(Ab) by using projective resolutions. It is however hard to find any explicit resolutions by projective condensed abelian groups, as extremally disconnected profinite sets are very fragile objects – nearly any operation done to extremally disconnected profinite sets will lead to an object that is not extremally disconnected, for example the product $S_1 \times S_2$ of two infinite extremally disconnected profinite sets S_1, S_2 is never extremally disconnected.[3]

In practice, however, $\mathbb{Z}[S]$ behaves like a projective object for any profinite set S. To quantify this, set

$$H^i(S, M) = \mathrm{Ext}^i_{\mathrm{Cond(Ab)}}(\mathbb{Z}[S], M)$$

for any condensed abelian group M. (This agrees with the derived functor of $M \mapsto M(S)$.) The following general proposition holds true more generally on any coherent site.

[3] Let us however mention one nontrivial stability property: A theorem of Vermeer [Ver95] says that for any endomorphism $f : S \to S$ of an extremally disconnected profinite set S, the fixed point set S^f is itself extremally disconnected.

Proposition 2.10. *Let*

$$M : \mathrm{ProFin}^{\mathrm{op}} \to \mathrm{Ab}$$

be a functor, i.e. a presheaf of abelian groups on ProFin*. Assume that M preserves finite products, and that for any surjective map $f : T \to S$, the complex*

$$0 \to M(S) \to M(T) \to M(T \times_S T) \to M(T \times_S T \times_S T) \to \dots$$

is exact.

Then M is a condensed abelian group, and for all profinite sets S and $i > 0$, one has $H^i(S, M) = 0$ for $i > 0$.

Proof. This has not been formalized yet, so we indicate the proof. Proposition 2.3 shows that M is a condensed abelian group. We prove by induction on $i > 0$ that $H^i(S, M) = 0$ for all profinite sets S, so assume the vanishing of $\mathrm{Ext}^1, \dots, \mathrm{Ext}^i$ for some $i \geq 0$. (This is vacuous for $i = 0$.) We aim to prove that $H^{i+1}(S, M) = 0$ for all profinite sets S. Pick any profinite set S and a cover $T \to S$ with $T \in \mathrm{ExtrDisc}$. We get a long exact sequence of condensed abelian groups

$$\dots \to \mathbb{Z}[T \times_S T \times_S T] \to \mathbb{Z}[T \times_S T] \to \mathbb{Z}[T] \to \mathbb{Z}[S] \to 0 :$$

Indeed, taken as presheaves on ExtrDisc, this is already true on the level of presheaves, where it reduces to the case of surjections of sets in which case one can write down a contracting homotopy. (Actually, the similar result is true in any topos, where one has to maybe argue a bit more carefully.)

The following argument is making explicit something usually seen through a spectral sequence. Define inductively

$$K_1 = \ker(\mathbb{Z}[T] \to \mathbb{Z}[S]),$$
$$K_2 = \ker(\mathbb{Z}[T \times_S T] \to \mathbb{Z}[T])$$

etc. One gets exact sequences

$$0 \to K_n \to \mathbb{Z}[T^{n/S}] \to K_{n-1} \to 0$$

for $n \geq 2$. From the long exact sequence

$$\dots \to H^i(T, M) \to \mathrm{Ext}^i(K_1, M) \to H^{i+1}(S, M) \to H^{i+1}(T, M) = 0$$

we see that we have to prove that $\mathrm{Ext}^i(K_1, M) = 0$ (if $i > 0$, otherwise that $M(T)$ surjects onto $\mathrm{Hom}(K_1, M)$). Assuming $i > 0$, we can go on, and using the inductive hypothesis applied to the fibre products $T^{*/S}$, we inductively see that

$$H^{i+1}(S, M) = \mathrm{Ext}^i(K_1, M) = \mathrm{Ext}^{i-1}(K_2, M) = \dots = \mathrm{Ext}^1(K_i, M)$$

and eventually that this is the same as the cokernel of

$$M(T^{i/S}) \to \mathrm{Hom}(K_{i+1}, M).$$

But there is an exact sequence

$$0 \to \mathrm{Hom}(K_{i+1}, M) \to M(T^{(i+1)/S}) \to \mathrm{Hom}(K_{i+2}, M)$$

and $\mathrm{Hom}(K_{i+2}, M)$ injects into $M(T^{(i+2)/S})$. We see that

$$\mathrm{Hom}(K_{i+1}, M) = \ker(M(T^{(i+1)/S}) \to M(T^{(i+2)/S}))$$

and we need to see that

$$M(T^{i/S}) \to \mathrm{Hom}(K_{i+1}, M) = \ker(M(T^{(i+1)/S}) \to M(T^{(i+2)/S}))$$

is surjective, which is precisely the exactness of the Čech complex. □

An important case is that of complete normed group. In a normed group, we always assume that the triangle inequality $||m + m'|| \leq ||m|| + ||m'||$ is satisfied, and $||-m|| = ||m||$.

Proposition 2.11. *Let $(M, ||\cdot||)$ be a complete normed group, regarded as a topological group. Then the corresponding condensed abelian group $\underline{M}$ sends any profinite set S to the completion of normed group of locally constant maps $S \to M$ (with the supremum norm), and for any profinite set S, one has $H^i(S, \underline{M}) = 0$ for $i > 0$.*

Proof. This follows Proposition 2.10 and the part of [Sch20, Proposition 8.19] that is already formalized. □

In particular, this applies to the p-Banach spaces of Theorem 1.1. For $0 < p \leq 1$, a p-Banach space V is a topological vector space that is complete and whose topology is induced by a p-norm $||\cdot|| : V \to \mathbb{R}_{\geq 0}$, that is a norm satisfying the scaling behaviour $||rv|| = |r|^p||v||$ for $r \in \mathbb{R}$, $v \in V$. Any Banach space $(V, ||\cdot||)$ is a p-Banach space, for the norm $||\cdot||^p$.

In fact, we will often regard $\mathbb{R}$ as a $\mathbb{Z}[T^{\pm 1}]$-algebra via the map sending T to $\frac{1}{2}$. Then we can more generally, for $1 > r \geq \frac{1}{2}$, consider r-Banach $\mathbb{Z}[T^{\pm 1}]$-modules: These are complete normed $\mathbb{Z}[T^{\pm 1}]$-modules M whose norm satisfies $||Tm|| = r||m||$ for $m \in M$. Then a p-Banach V is in particular an r-Banach $\mathbb{Z}[T^{\pm 1}]$-module, where $r = 2^{-p}$.

One can also complete $\mathbb{Z}[T^{\pm 1}]$ itself with respect to the r-norm

$$||\sum_{n\in\mathbb{Z}} a_n T^n|| = \sum_{n\in\mathbb{Z}} |a_n| r^n,$$

leading to a ring of arithmetic Laurent series $\mathbb{Z}((T))_r$ that converge on some disc $\{0 < |T| \leq r\} \subset \mathbb{C}$. This ring will actually play a key role in the proof of Theorem 1.1.

3. Spaces of measures

At this point, we have discussed the abelian category Cond(Ab) and the notion of p-Banach spaces that appears. It remains to define the space of measures $\mathcal{M}_{p'}(S)$. In order to avoid clutter of notation, we rename p' to p in this section.

Let us first give an informal description, working here with topological vector spaces. The space $\mathcal{M}_p(S)$ of p-measures on S is a certain subspace of the space of (signed Radon) measures $\mathcal{M}(S)$ on S. The easiest definition of the latter is as the dual

$$\mathcal{M}(S) = \mathrm{Hom}(C(S, \mathbb{R}), \mathbb{R})$$

to the space of continuous functions on S. We equip $\mathcal{M}(S)$ with the compactly generated version of the weak topology; in other words, $\mathcal{M}(S)$ is written as the union

$$\mathcal{M}(S) = \bigcup_{c>0} \mathcal{M}(S)_{\leq c}$$

of the subspaces of measures of norm $\leq c$, each of which is naturally a compact Hausdorff space in the weak topology. Now for $0 < p < 1$, we let

$$\mathcal{M}_p(S) \subset \mathcal{M}(S)$$

be the subspace of those measures that can be written as sums

$$\sum_{n\geq 0} r_n[s_n]$$

for a sequence of elements $s_n \in S$ and $r_n \in \mathbb{R}$ with $\sum_{n\geq 0} |r_n|^p < \infty$. The subspace

$$\mathcal{M}_p(S)_{\leq c} \subset \mathcal{M}_p(S) \subset \mathcal{M}(S)$$

of those measures where $\sum_{n\geq 0} |r_n|^p \leq c$ is closed in $\mathcal{M}(S)_{\leq c}$, and thus itself compact Hausdorff. Then

$$\mathcal{M}_p(S) = \bigcup_{c>0} \mathcal{M}_p(S)$$

is given the colimit topology.

Actually, a better way to describe these spaces as condensed vector spaces is the following; this is also very close to the description of $\mathbb{Z}[S]$ in Proposition 2.6. The following recovers $\mathcal{M}(S)$ by taking $p = 1$.

Definition 3.1. Let $0 < p \leq 1$, and let $S = \varprojlim_i S_i$ be a profinite set written as a cofiltered limit of finite sets. For any $c > 0$, let

$$\mathbb{R}[S_i]_{\ell^p \leq c} \subset \mathbb{R}[S_i]$$

be the closed subset of all $\sum_{s\in S_i} r_s[s]$ with $\sum_{s\in S_i} |r_s|^p \leq c$. Let

$$\mathcal{M}_p(S)_{\leq c} = \varprojlim_i \mathbb{R}[S_i]_{\ell^p \leq c}$$

which is naturally a compact Hausdorff space, and

$$\mathcal{M}_p(S) = \bigcup_{c>0} \mathcal{M}_p(S)_{\leq c}.$$

We will not need to know the real vector space structure. The structure that we need is that this is a compact-Hausdorff-filtered object with an abelian group structure.

Definition 3.2. A compact-Hausdorffly-filtered-pseudonormed abelian group is an abelian group X together with negation-stable subsets $X_{\leq c} \subset X$ for all $c > 0$, exhausting X, and compact Hausdorff topologies on all $X_{\leq c}$ such that negation is continuous, $0 \in X_{\leq c}$ for all c and the addition defines continuous maps

$$X_{\leq c} \times X_{\leq c'} \to X_{\leq c+c'}.$$

We note that there are two category structures one can consider: Those maps that strictly preserve the (pseudo)norm, or those that only preserve it up to multiplication by a scalar. Both structures are relevant. Regarding strict maps, we have inverse limits.

Proposition 3.3. *Consider an inverse system $(X_i)_i$ of compact-Hausdorffly-filtered-pseudonormed abelian groups where all transition maps $X_i \to X_j$ send $X_{i,\leq c}$ to $X_{j,\leq c}$. Then*

$$X_{\leq c} := \varprojlim_i X_{i,\leq c}$$

is compact Hausdorff, and

$$X = \bigcup_c X_{\leq c}$$

is naturally a compact-Hausdorffly-filtered pseudonormed abelian group which is the limit of $(X_i)_i$ in the strict category structure.

Proof. One can define negation and addition on X as continuous maps $- : X_{\leq c} \to X_{\leq c}$ and $+ : X_{\leq c} \times X_{\leq c'} \to X_{\leq c+c'}$, and these pass to the unions. It should then be straightforward to check the axioms. □

Any such object naturally gives rise to a condensed abelian group

$$\underline{X} = \bigcup_c \underline{X_{\leq c}}.$$

This is functorial in non-strict maps. We note the following exactness property.

Proposition 3.4. *Consider a short exact sequence of abelian groups*

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$$

such that all of X', X and X'' carry the structure of compact-Hausdorffly-filtered-pseudonormed abelian groups. Assume that $f(X'_{\leq c}) \subset X_{\leq c}$ and $g(X_{\leq c}) \subset X''_{\leq c}$. Moreover, assume that there are some c_f and c_g so that $\ker(g) \cap X_{\leq c} \subset f(X'_{\leq c_f c})$ and $X''_{\leq c} \subset g(X_{\leq c_g c})$. Then the sequence

$$0 \to \underline{X'} \to \underline{X} \to \underline{X''} \to 0$$

of condensed abelian groups is exact.

Proof. We evaluate at $S \in \mathrm{ExtrDisc}$. Any map $S \to X''$ factors over some $X''_{\leq c}$, and then $g : X_{\leq c_g c} \times_{X''} X''_{\leq c} \to X''_{\leq c}$ is a surjection of compact Hausdorff spaces; as S is extremally disconnected, the map $S \to X''_{\leq c}$ can be lifted, showing that $g : X(S) \to X''(S)$ is surjective. A similar argument shows that the kernel of $g : X(S) \to X''(S)$ is in the image of $f : X'(S) \to X(S)$, the latter clearly being injective. □

It will also be useful to note that the hypothesis of Proposition 3.4 is stable under the limits of Proposition 3.3. The philosophical reason for the disappearance of $\varprojlim^1$-issues is the good behaviour of limits of compact Hausdorff spaces.

Proposition 3.5. *Consider an inverse system*

$$(0 \to X'_i \xrightarrow{f_i} X_i \xrightarrow{g_i} X''_i \to 0)_i$$

of short exact sequences as in Proposition 3.4, satisfying the hypothesis for fixed constants c_f and c_g. Moreover, assume that the transition maps $X'_i \to X'_j$, $X_i \to X_j$ and $X''_i \to X''_j$ are strict, and let X', X and X'' be their limits. Then

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$$

satisfies the hypotheses of Proposition 3.4 with the same c_f and c_g.

Proof. Pass to cofiltered limits of compact Hausdorff spaces in statements such as $X''_{i,\leq c} \subset g(X_{i,\leq c_g c})$, noting that cofiltered limits of surjections of compact Hausdorff spaces are still surjective (by an application of Tychonoff). □

For Theorem 1.1, we need to compute

$$\mathrm{Ext}^i(\mathcal{M}_{p'}(S), V).$$

The Ext-group here is computed by taking a projective resolution of the source. Recall that these projective generators are of the form $\mathbb{Z}[S]$ with S extremally disconnected. We are allowed to take more generally profinite S, by Proposition 2.11. However, we are *not* allowed to take compact Hausdorff S, as Proposition 2.11 will fail in general in that case, even for p-Banach's when $p < 1$.

Thus, we are forced to resolve a real vector space $\mathcal{M}_{p'}(S)$ by $\mathbb{Z}[S']$ for profinite S'. This seems problematic, and it is here that we make a crucial turn to arithmetic Laurent series $\mathbb{Z}((T))_r \to \mathbb{R}$.

Note that there is a short exact sequence

$$0 \to \mathbb{Z}((T))_r \xrightarrow{1-2T} \mathbb{Z}((T))_r \to \mathbb{R} \to 0.$$

In fact, we note that, up to changing the topology on $\mathbb{Z}((T))_r$ slightly(!), all terms here are naturally compact-Hausdorffly-filtered:

(1) On $\mathbb{Z}((T))_r$, we let

$$\mathbb{Z}((T))_{r,\leq c} = \{\sum_{n\in\mathbb{Z}} a_n T^n \mid \sum_{n\in\mathbb{Z}} |a_n| r^n \leq c\}.$$

Note that in particular each a_n is bounded in there, and for $n \ll 0$, $a_n = 0$ necessarily. This gives an embedding of $\mathbb{Z}((T))_{r,\leq c}$ into $\prod_{n\gg -\infty}[-c_n, c_n]$ for some constants c_n (depending on c and r). The latter is a profinite set, and $\mathbb{Z}((T))_{r,\leq c}$ is a closed subset. This makes $\mathbb{Z}((T))_{r,\leq c}$ itself into a profinite set, and then $\mathbb{Z}((T))_r$ into a compact-Hausdorffly-filtered pseudonormed abelian group.

(2) On $\mathbb{R}$, we let

$$\mathbb{R}_{\leq c} = \{x \in \mathbb{R} \mid |x|^p \leq c\}$$

where p is chosen so that $r = 2^{-p}$.

Proposition 3.6. *The sequence*

$$0 \to \mathbb{Z}((T))_r \xrightarrow{1-2T} \mathbb{Z}((T))_r \to \mathbb{R} \to 0.$$

satisfies the hypotheses of Proposition 3.4, for some explicit constants c_f, c_g.

Proof. This is a rather straightforward calculation, see [Sch20, Proposition 7.2]. The result has been formalized by Filippo A.E. Nuccio. □

Remark 3.7. This proposition translates the non-convex ℓ^p-norm on $\mathbb{R}$ into the ℓ^1-norm on $\mathbb{Z}((T))_r$ (note that the norm on $\mathbb{Z}((T))_r$ is given by the ℓ^1-norm $|a_n|$ on the coefficients). This is not a contradiction because of the discretization of the coefficients in $\mathbb{Z}((T))_r$, but we believe this passage from continuous non-convexity to "discrete convexity" is a key aspect of the proof.

We can now also define similar spaces of measures over $\mathbb{Z}((T))_r$. In fact, for any finite set S, we can make the free module

$$\mathbb{Z}((T))_r[S]$$

compact-Hausdorffly-filtered by the subsets

$$\mathbb{Z}((T))_r[S]_{\leq c} = \{\sum_{s\in S} f_s[s] \mid f_s \in \mathbb{Z}((T))_{r,\leq c_s}, \sum c_s \leq c\},$$

and then pass to limits for profinite sets S as in the definition of $\mathcal{M}_p(S)$, i.e. formally using Proposition 3.3:

$$\mathcal{M}(S, \mathbb{Z}((T))_r)_{\leq c} = \varprojlim_i \mathbb{Z}((T))_r[S_i]_{\leq c},$$

with union

$$\mathcal{M}(S, \mathbb{Z}((T))_r) = \bigcup_c \mathcal{M}(S, \mathbb{Z}((T))_r)_{\leq c}.$$

One can formally pass to free modules in Proposition 3.6, keeping the same constants, and then Proposition 3.5 and Proposition 3.4 imply the following corollary.

Corollary 3.8. *There is a short exact sequence of condensed abelian groups*

$$0 \to \mathcal{M}(S, \mathbb{Z}((T))_r) \xrightarrow{1-2T} \mathcal{M}(S, \mathbb{Z}((T))_r) \to \mathcal{M}_p(S) \to 0.$$

In particular, $1 - 2T$ is a nonzerodivisor on $\mathcal{M}(S, \mathbb{Z}((T))_r)$, and

$$\mathcal{M}(S, \mathbb{Z}((T))_r)/(1 - 2T) \cong \mathcal{M}_p(S).$$

For later reference, we note that we could replace $1 - 2T$ by $T^{-1} - 2$ here, as T is a unit.

We can now reformulate Theorem 1.1 in terms of $\mathcal{M}(S, \mathbb{Z}((T))_r)$. The following is [Sch20, Theorem 9.1].

Theorem 3.9. *Let $1 > r' > r > 0$ be real numbers, let S be a profinite set, and let V be an r-Banach module over $\mathbb{Z}[T^{\pm 1}]$. Then*

$$\mathrm{Ext}^i_{\mathbb{Z}[T^{-1}]}(\mathcal{M}(S, \mathbb{Z}((T))_{r'}), V) = 0$$

for $i > 0$.

As T^{-1} acts as an isomorphism on both terms, we could also compute the Ext-groups over $\mathbb{Z}[T^{\pm 1}]$, or over $\mathbb{Z}[T]$; however, this formulation turns out to be most convenient in the proof.

Remark 3.10. Regarding the formalization, we note that the statement can be reformulated without appeal to Ext-groups over $\mathbb{Z}[T^{-1}]$ by using the resolution

$$0 \to \mathcal{M}(S, \mathbb{Z}((T))_{r'})[T^{-1}] \xrightarrow{T^{-1}-[T^{-1}]} \mathcal{M}(S, \mathbb{Z}((T))_{r'})[T^{-1}] \to \mathcal{M}(S, \mathbb{Z}((T))_{r'}) \to 0$$

which induces a long exact sequence

$$\ldots \to \mathrm{Ext}^i_{\mathbb{Z}[T^{-1}]}(\mathcal{M}(S,\mathbb{Z}((T))_{r'}),V) \to \mathrm{Ext}^i(\mathcal{M}(S,\mathbb{Z}((T))_{r'}),V)$$
$$\xrightarrow{(T^{-1})_V-(T^{-1})_{\mathcal{M}}} \mathrm{Ext}^i(\mathcal{M}(S,\mathbb{Z}((T))_{r'}),V) \to \mathrm{Ext}^{i+1}_{\mathbb{Z}[T^{-1}]}(\mathcal{M}(S,\mathbb{Z}((T))_{r'}),V) \to \ldots.$$

In other words, Theorem 3.9 can be formulated as the bijectivity of

$$(T^{-1})_V - (T^{-1})_{\mathcal{M}} : \mathrm{Ext}^i(\mathcal{M}(S,\mathbb{Z}((T))_{r'}),V) \to \mathrm{Ext}^i(\mathcal{M}(S,\mathbb{Z}((T))_{r'}),V)$$

for $i > 0$, and its surjectivity for $i = 0$.

The advantage of this reformulation is that now $\mathcal{M}(S,\mathbb{Z}((T))_{r'})$ has filtration steps that are profinite (as opposed to merely compact Hausdorff), so the task of resolving by $\mathbb{Z}[S']$'s with profinite S' seems more manageable. It turns out that another advantage of the formulation of Theorem 3.9 is that the scaling behaviour of the integers is decoupled from the scaling behaviour of the independent variable T. (In $\mathbb{R}$, they are of course intertwined, as $T^{-1} = 2$.)

Let us first explain how Theorem 3.9 implies Theorem 1.1.

Theorem 3.9 implies Theorem 1.1. Given $0 < p' < p \leq 1$, let $r = 2^{-p}$ and $r' = 2^{-p'}$; then $1 > r' > r > 0$, and V becomes an r-Banach module via restriction of scalars along $\mathbb{Z}[T^{\pm 1}] \to \mathbb{R} : T \mapsto \frac{1}{2}$. Thus, Theorem 3.9 gives the vanishing of

$$\mathrm{Ext}^i_{\mathbb{Z}[T^{-1}]}(\mathcal{M}(S,\mathbb{Z}((T))_{r'}),V) = 0$$

for $i > 0$.

Using the reformulation of Theorem 3.9 as Remark 3.10, we note that the T^{-1}-action on V is given by multiplication by 2, so this can be translated into the bijectivity of

$$2 - T^{-1} : \mathrm{Ext}^i(\mathcal{M}(S,\mathbb{Z}((T))_{r'}),V) \to \mathrm{Ext}^i(\mathcal{M}(S,\mathbb{Z}((T))_{r'}),V)$$

and its surjectivity for $i = 0$. Now the long exact sequence coming from Corollary 3.8 (applied with r' in place of r) gives Theorem 1.1. □

As a final reduction in this section, we have the following proposition.

Proposition 3.11. *Decomposing $\mathbb{Z}((T))_r$ into positive and nonpositive coefficients yields a direct sum decomposition*

$$\mathbb{Z}((T))_r = T\mathbb{Z}[[T]]_r \oplus \mathbb{Z}[T^{-1}].$$

This extends to a decomposition of spaces of measures

$$\mathcal{M}(S,\mathbb{Z}((T))_r) = \mathcal{M}(S,T\mathbb{Z}[[T]]_r) \oplus \mathcal{M}(S,\mathbb{Z}[T^{-1}])$$

where $\mathcal{M}(S,\mathbb{Z}[T^{-1}]) = \mathbb{Z}[T^{-1}][S]$ is the free condensed $\mathbb{Z}[T^{-1}]$-module on S. Letting $\overline{\mathcal{M}}_r(S) = \mathcal{M}(S,T\mathbb{Z}[[T]]_r)$, we get a short exact sequence of condensed $\mathbb{Z}[T^{-1}]$-modules

$$0 \to \mathbb{Z}[T^{-1}][S] \to \mathcal{M}(S,\mathbb{Z}((T))_r) \to \overline{\mathcal{M}}_r(S) \to 0.$$

Proof. On $\mathbb{Z}((T))_{r,\leq c}$, only finitely many nonpositive coefficients can possibly be nonzero, and each of them is bounded. This shows that the nonpositive summand of $\mathbb{Z}((T))_r$ is given by $\mathbb{Z}[T^{-1}]$. To pass to profinite S, use Proposition 2.6. □

Using the long exact sequence and Proposition 2.11 (and the equality $\mathrm{Ext}^i_{\mathbb{Z}[T^{-1}]}(\mathbb{Z}[T^{-1}][S], M) = \mathrm{Ext}^i(\mathbb{Z}[S], M)$, whose precise proof will depend on the formalization chosen to talk about Ext-groups over $\mathbb{Z}[T^{-1}]$), Theorem 3.9 follows from the following:

Theorem 3.12. *Let $1 > r' > r > 0$, let S be a profinite set, and let V be an r-Banach $\mathbb{Z}[T^{\pm 1}]$-module. Then for all $i \geq 0$,*

$$\mathrm{Ext}^i_{\mathbb{Z}[T^{-1}]}(\overline{\mathcal{M}}_{r'}(S), V) = 0.$$

Again, for formalization purposes it may be profitable to rephrase this as in Remark 3.10.

4. MacLane's Q'-construction

The proof in [Sch20] now proceeds by using Breen–Deligne resolutions. These give a resolution of a condensed abelian group A of the form

$$\ldots \to \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0$$

but the resolution is inexplicit – one only knows that it can be continued in a way where each term is a finite direct sum of some $\mathbb{Z}[A^n]$'s, and all differentials are given by some universal formulas. It is slightly surprising that one can make explicit computations of Ext-groups by using an inexplicit resolution; the reason it works is that Theorem 3.12 is asking about the vanishing of all Ext-groups, which is a qualitative statement, and thus qualitative knowledge of a resolution can be good enough.

Unfortunately, the existence of a Breen–Deligne resolution needs some basic results in stable homotopy theory (cf. [Sch19, Appendix to Lecture 4]), specifically about the homology of Eilenberg–MacLane spaces. (The knowledge required is closely related to the finiteness of stable homotopy groups of spheres.)

However, in the course of working on the formalization of [Sch20, Theorem 9.4], the first author realized that there is a very explicit complex having many of the same properties as a Breen–Deligne resolution (while not actually being a resolution). Later, we realized that this complex had been constructed many years ago by MacLane, who called it Q'. See for example [Mac57, §4] for an explicit description of Q', or one of [EM51, §12] and [Mac58, §3] (where the construction is denoted Q).

Definition 4.1. The MacLane Q'-construction is the unique functorial association taking an abelian group A to the complex

$$\ldots \xrightarrow{d_{A,n}} \mathbb{Z}[A^{2^n}] \xrightarrow{d_{A,n-1}} \ldots \xrightarrow{d_{A,2}} \mathbb{Z}[A^4] \xrightarrow{d_{A,1}} \mathbb{Z}[A^2] \xrightarrow{d_{A,0}} \mathbb{Z}[A]$$

with the property that the identity maps $\mathbb{Z}[A^{2^n}] \to \mathbb{Z}[A^{2^n}]$ define a homotopy between the two maps from

$$\ldots \xrightarrow{d_{A^2,n}} \mathbb{Z}[A^{2^{n+1}}] \xrightarrow{d_{A^2,n-1}} \ldots \xrightarrow{d_{A^2,2}} \mathbb{Z}[A^8] \xrightarrow{d_{A^2,1}} \mathbb{Z}[A^4] \xrightarrow{d_{A^2,0}} \mathbb{Z}[A^2]$$

to

$$\ldots \xrightarrow{d_{A,n}} \mathbb{Z}[A^{2^n}] \xrightarrow{d_{A,n-1}} \ldots \xrightarrow{d_{A,2}} \mathbb{Z}[A^4] \xrightarrow{d_{A,1}} \mathbb{Z}[A^2] \xrightarrow{d_{A,0}} \mathbb{Z}[A]$$

given by the addition map $A^2 \to A$, respectively by the sum of the two maps induced by the projection maps $A^2 \to A$.

We leave it to the reader to figure out that the condition allows one to define the differentials inductively in a unique way. The first differential, in particular, has to be given by the map $\mathbb{Z}[A^2] \to \mathbb{Z}[A] : [(a,b)] \mapsto [a+b] - [a] - [b]$. One easily sees that $H_0(Q'(A)) \cong A$ naturally in A.

Modulo the computation of $Q'(\mathbb{Z})$, one can compute the homology of $Q'(A)$ for any A, at least if A is torsion-free. (In the general case, a similar statement holds true, but involves Tor-groups.)

Proposition 4.2. *For any $i \geq 0$, the functor $A \mapsto H_i(Q'(A))$ has the following properties:*

(1) It is additive, i.e.

$$H_i(Q'(A \oplus B)) \cong H_i(Q'(A)) \oplus H_i(Q'(B)).$$

(2) It commutes with filtered colimits, i.e. for a filtered inductive system A_i,

$$\varinjlim_i H_i(Q'(A)) \cong H_i(Q'(\varinjlim_i A_i)).$$

In particular, for torsion-free abelian groups A, there is a functorial isomorphism

$$H_i(Q'(A)) \cong H_i(Q'(\mathbb{Z})) \otimes A.$$

As the proof shows, we do not really need the Q'-construction here: Any "Breen–Deligne package" in the sense of the formalization of [Sch20, Theorem 9.4] will do.

Proof. Let us do the easy things first. Part (2) is clear as everything in sight commutes with filtered colimits. Assuming (1), we note that there is a natural map

$$H_i(Q'(\mathbb{Z})) \times A \to H_i(Q'(A))$$

induced by functoriality of $H_i(Q'(-))$. To check that this is bilinear and induces an isomorphism

$$H_i(Q'(\mathbb{Z})) \otimes A \cong H_i(Q'(A)),$$

we can reduce to the case that A is finitely generated by (2). In that case A is finite free, and the result follows from (1).

Thus, it remains to prove part (1), which has already been formalized. We recall that the direct sum of two abelian groups M and N is characterized as the abelian group P with maps $i_M : M \to P$, $i_N : N \to P$, $p_M : P \to M$, $p_N : P \to N$, satisfying $p_M i_M = \mathrm{id}_M$, $p_N i_N = \mathrm{id}_N$, $p_M i_N = 0$, $p_N i_M = 0$, $\mathrm{id}_P = i_M p_M + i_N p_N$. Apply this to $M = H_i(Q'(A))$, $N = H_i(Q'(B))$ and $P = H_i(Q'(A \oplus B))$, with all maps induced by applying $H_i(Q'(-))$ to the similar maps for A, B and $A \oplus B$. The fact that $H_i(Q'(-))$ is a functor already gives all identities except $\mathrm{id}_P = i_M p_M + i_N p_N$, and the only issue is the question whether $H_i(Q'(-))$ induces additive maps on morphism spaces. But if $f, g : C \to D$ are any two maps of abelian groups, then $H_i(Q'(f+g)) = H_i(Q'(f)) + H_i(Q'(g))$, by reducing to the universal case of the two projections $D^2 \to D$ and using the homotopy baked into Definition 4.1. □

Remark 4.3. Thinking ∞-categorically, the Q'-construction is determined by an additive functor from finite free $\mathbb{Z}$-modules to $\mathcal{D}_{\geq 0}(\mathbb{Z})$. Any such functor is, by Morita theory, given by $A \mapsto A \otimes_{\mathbb{Z}} M$ for some $M \in \mathcal{D}_{\geq 0}(\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z})$; here $\mathbb{S}$ denotes the sphere spectrum. In fact, one can show that

$$M = \bigoplus_{i \geq 0} (\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z})^{2^i}[i],$$

so

$$Q'(A) \cong \bigoplus_{i\geq 0} (A \otimes_{\mathbb{S}} \mathbb{Z})^{2^i}[i].$$

As we will have no use for these results here, let us content ourselves with noting that this can be easily deduced from Goodwillie calculus.

We note that by functoriality of the Q'-construction, it can also be applied to condensed abelian groups.

Corollary 4.4. *For torsion-free condensed abelian groups A, there is a natural isomorphism*

$$H_i(Q'(A)) \cong H_i(Q'(\mathbb{Z})) \otimes A$$

of condensed abelian groups.

Remark 4.5. Here, we only need to be able to tensor condensed abelian groups with (abstract) abelian groups. (With more effort, one could prove that $H_i(Q'(\mathbb{Z}))$ is even finitely generated.) In that case, the tensor product functor can be defined very naively by tensoring the values at any S with the given abstract abelian group.

Proof. Evaluating at $S \in \mathrm{ExtrDisc}$, we note that $S \mapsto H_i(Q'(A(S)))$ is already a condensed abelian group, and agrees with $H_i(Q'(\mathbb{Z})) \otimes A(S)$. Thus, the same is true after sheafification. □

If A is a torsion-free condensed abelian group equipped with an endomorphism f, then $Q'(A)$ is also equipped with the endomorphism f induced by functoriality, and by functoriality all previous assertions upgrade to $\mathbb{Z}[f]$-modules. We will need the following proposition in the proof of Theorem 3.12.

Proposition 4.6. *Let M and N be condensed abelian groups with endomorphisms f_M, f_N. Assume that M is torsion-free (over $\mathbb{Z}$). Then*

$$\mathrm{Ext}^i_{\mathbb{Z}[f]}(M, N) = 0$$

for all $i \geq 0$ if and only if

$$\mathrm{Ext}^i_{\mathbb{Z}[f]}(Q'(M), N) = 0$$

for all $i \geq 0$. More precisely, the first vanishes for $0 \leq i \leq j$ if and only if the second vanishes for $0 \leq i \leq j$.

At this point, we need to be able to talk about Ext-groups of (bounded to the right) complexes of condensed abelian groups (against condensed abelian groups).

The statement is also true without the torsion-freeness assumption on M, but slightly more nasty to prove then (and not required for the application).

Proof. We induct on j. Consider first the case $j = 0$; then any map $Q'(M) \to N$ factors uniquely over $H_0 Q'(M)[0] = M[0]$, yielding the result. Now assume that both sides vanish for $0 \leq i < j$; we need to see that the vanishing of the Ext^i's is equivalent. Consider the triangle

$$\tau_{\geq 1} Q'(M) \to Q'(M) \to M[0] \to .$$

Taking the corresponding long exact sequence of Ext-groups against N, we see that it suffices to see that

$$\mathrm{Ext}^i_{\mathbb{Z}[f]}(\tau_{\geq 1}Q'(M), N) = 0$$

for $0 \leq i \leq j$. But we can prove by descending induction on t that

$$\mathrm{Ext}^i_{\mathbb{Z}[f]}(\tau_{\geq t}Q'(M), N) = 0.$$

This is trivially true for $t > i$. Now look at the triangle

$$\tau_{\geq t+1}Q'(M) \to \tau_{\geq t}Q'(M) \to H_t(Q'(M))[t] \to$$

and the corresponding long exact sequence. It becomes sufficient to prove that

$$\mathrm{Ext}^i_{\mathbb{Z}[f]}(H_t(Q'(M))[t], N) = 0$$

for $0 \leq i \leq j$. Trivially,

$$\mathrm{Ext}^i_{\mathbb{Z}[f]}(H_t(Q'(M))[t], N) = \mathrm{Ext}^{i-t}_{\mathbb{Z}[f]}(H_t(Q'(M)), N).$$

Note that $t \geq 1$ here, so $i - t < j$ (and can be assumed ≥ 0). Also $H_t(Q'(M)) \cong H_t(Q'(\mathbb{Z})) \otimes M$. Thus, it suffices to show that for every abelian group A and every $0 \leq i < j$,

$$\mathrm{Ext}^i_{\mathbb{Z}[f]}(A \otimes M, N) = 0.$$

If A is free, then $A \otimes M$ is a direct sum of copies of M, and the result follows as Ext turns direct sums into products (and we assumed the vanishing of $\mathrm{Ext}^i_{\mathbb{Z}[f]}(M, N)$ for $0 \leq i < j$). In general, one can pick a two-term free resolution of A and use the long exact sequence. $\square$

5. Final reduction

With Proposition 4.6, Theorem 3.12 reduces to the following assertion. Pick $1 > r' > r > 0$, a profinite S, and some r-Banach $\mathbb{Z}[T^{\pm 1}]$-module V as before. Then we want to prove that

$$\mathrm{Ext}^i_{\mathbb{Z}[T^{-1}]}(Q'(\overline{\mathcal{M}}_{r'}(S)), V) = 0$$

for all $i \geq 0$.

At this point, it is profitable to rewrite this again as the bijectivity of

$$(T^{-1})_V - (T^{-1})_{\mathcal{M}} : \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S)), V) \to \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S)), V).$$

Now these Ext-groups can be computed! More precisely, recall that $Q'(\overline{\mathcal{M}}_{r'}(S))$ is a complex of the form

$$\ldots \to \mathbb{Z}[\overline{\mathcal{M}}_{r'}(S)^2] \to \mathbb{Z}[\overline{\mathcal{M}}_{r'}(S)] \to 0.$$

Termwise, the Ext-groups turn into cohomology groups

$$H^i(\overline{\mathcal{M}}_{r'}(S)^{2^j}, V).$$

Unfortunately, $\overline{\mathcal{M}}_{r'}(S)$ itself is not profinite, so we cannot directly apply Proposition 2.11. To get around this last cliff, we write $Q'(\overline{\mathcal{M}}_{r'}(S))$ as a filtered colimit of complexes

$$Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq c} : \ldots \to \mathbb{Z}[\overline{\mathcal{M}}_{r'}(S)^2_{\leq \kappa_1 c}] \to \mathbb{Z}[\overline{\mathcal{M}}_{r'}(S)_{\leq \kappa_0 c}] \to 0$$

where the constants $\kappa_0 = 1, \kappa_1, \ldots$ are positive and chosen so that all differentials are well-defined. (The possibility of choosing such constants has already been formalized, as part of the formalization of [Sch20, Theorem 9.4].) It suffices to prove that

$$(T^{-1})_V - (T^{-1})_{\mathcal{M}} : \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq c}, V) \to \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq r'c}, V)$$

is a pro-isomorphism in c, as then the final result follows by passing to a derived limit over c, see Lemma 5.1 below. This final pro-isomorphism assertion can finally be written out, and it unravels to the statement of [Sch20, Theorem 9.4] that has been formalized.

In passing to the derived limit over c, we use the following lemma.

Lemma 5.1. *Assume that in each degree i, the map*

$$(T^{-1})_V - (T^{-1})_{\mathcal{M}} : \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq c}, V) \to \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq r'c}, V)$$

is a pro-isomorphism in c (i.e., pro-systems of kernels, and of cokernels, are pro-zero). Then

$$(T^{-1})_V - (T^{-1})_{\mathcal{M}} : \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S)), V) \to \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S)), V).$$

is an isomorphism.

Proof. We have

$$Q'(\overline{\mathcal{M}}_{r'}(S)) = \bigcup_n Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq n},$$

inducing a resolution

$$0 \to \bigoplus_n Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq n} \to \bigoplus_n Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq n} \to Q'(\overline{\mathcal{M}}_{r'}(S)) \to 0.$$

Passing to a corresponding long exact sequence reduces one to checking that the squares

$$\begin{array}{ccc} \prod_n \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq n}, V) & \longrightarrow & \prod_n \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq n}, V) \\ \downarrow & & \downarrow \\ \prod_n \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq n}, V) & \longrightarrow & \prod_n \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq n}, V) \end{array}$$

are bicartesian (here, horizontal maps are shift minus identity, and vertical maps are $(T^{-1})_V - (T^{-1})_{\mathcal{M}}$). Equivalently, the horizontal maps become isomorphisms on vertical kernels, and vertical cokernels. But the vertical kernels and vertical cokernels induce pro-zero systems of abelian groups, and then the horizontal kernels and cokernels compute $\varprojlim_n$ and $\varprojlim^1_n$ of their systems, which vanish. □

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