GRONDSLAGEN VAN DE WISKUNDE 24/25 — HOMEWORK

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1. Homework 1

(1) (2pts) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two functions.
  (a) Prove that if $g \circ f$ is surjective, then g is surjective.
  (b) Prove that if $g \circ f$ is injective, then f is injective.

(2) (2pts) Let $f\colon X \to Y$ be a function. Assume that f has a *left inverse* $g\colon Y \to X$, and a *right inverse* $h\colon Y \to X$. (In other words, $g \circ f = \mathrm{id}_X$ and $f \circ h = \mathrm{id}_Y$.) Prove that $g = h$.

(3) (3pts) Prove that $2^{|\mathbb{N}|} \times 2^{|\mathbb{N}|} = 2^{|\mathbb{N}|}$.

(4) (3pts) Prove that $|\mathbb{N}^{\mathbb{N}}| = 2^{|\mathbb{N}|}$.

2. Homework 2

The purpose of this homework is to prove the existence of nonprincipal ultrafilters. We will return to this topic in Section 2 in a few weeks.

Let X be a set. A *filter* on X is a collection $\mathcal{F}$ of subsets of X satisfying the following conditions:

(F1) $X \in \mathcal{F}$.
(F2) If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.
(F3) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

(NB: Definition 2.5.6 in the book also requires $\varnothing \notin \mathcal{F}$, but we do not impose this condition!) Since filters are subsets of the powerset $\mathcal{P}(X)$, we can compare filters via the inclusion relation on $\mathcal{P}(X)$.

A filter $\mathcal{F}$ is *proper* if it is not equal to the powerset $\mathcal{P}(X)$, or equivalently, if $\varnothing \notin \mathcal{F}$. A filter is called an *ultrafilter* if it is a maximal proper filter for the inclusion relation.

(1) (1pt) Let A be subset of X. Show that the collection $\{B \subseteq X \mid A \subseteq B\}$ is a filter. This is the filter *generated by* A. Such filters are also called *principal*.

(2) (1pt) Let $\mathcal{C}$ be the collection $\{B \subseteq X \mid X - B \text{ is finite}\}$. Show that $\mathcal{C}$ is a filter. This filter is called the *cofinite filter*.

(3) (3pt) Let $\mathcal{F}$ be a proper filter on X. Use Zorn's lemma to prove that there exists an ultrafilter on X that contains $\mathcal{F}$.

(4) (2pt) Let $\mathcal{F}$ be a proper filter on X and suppose that A is a subset of X satisfying the following condition: For all $B \in \mathcal{F}$, the intersection $A \cap B$ is nonempty.
  Show that there exists a proper filter $\mathcal{F}'$ such that $A \in \mathcal{F}'$ and $\mathcal{F} \subseteq \mathcal{F}'$.

(5) (1pt) Let $\mathcal{U}$ be an ultrafilter and A a subset of X. Show that $A \in \mathcal{U}$ or $X - A \in \mathcal{U}$.

(6) (1pt) Show that a principal ultrafilter on X is generated by $\{x\}$ for some $x \in X$.

(7) (1pt) Assume that X is infinite. Show that the cofinite filter $\mathcal{C}$ is proper and not contained in any principal ultrafilter. Conclude that there exist ultrafilters that are not principal.