GRONDSLAGEN VAN DE WISKUNDE 24/25 — HOMEWORK

JOHAN COMMELIN

1. Homework 1

(1) (2pts) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two functions.
   (a) Prove that if $g \circ f$ is surjective, then g is surjective.
   (b) Prove that if $g \circ f$ is injective, then f is injective.

(2) (2pts) Let $f\colon X \to Y$ be a function. Assume that f has a *left inverse* $g\colon Y \to X$, and a *right inverse* $h\colon Y \to X$. (In other words, $g \circ f = \mathrm{id}_X$ and $f \circ h = \mathrm{id}_Y$.) Prove that $g = h$.

(3) (3pts) Prove that $2^{|\mathbb{N}|} \times 2^{|\mathbb{N}|} = 2^{|\mathbb{N}|}$.

(4) (3pts) Prove that $|\mathbb{N}^{\mathbb{N}}| = 2^{|\mathbb{N}|}$.

2. Homework 2

The purpose of this homework is to prove the existence of nonprincipal ultrafilters. We will return to this topic in Section 2 in a few weeks.

Let X be a set. A *filter* on X is a collection $\mathcal{F}$ of subsets of X satisfying the following conditions:

(F1) $X \in \mathcal{F}$.
(F2) If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.
(F3) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

(NB: Definition 2.5.6 in the book also requires $\varnothing \notin \mathcal{F}$, but we do not impose this condition!) Since filters are subsets of the powerset $\mathcal{P}(X)$, we can compare filters via the inclusion relation on $\mathcal{P}(X)$.

A filter $\mathcal{F}$ is *proper* if it is not equal to the powerset $\mathcal{P}(X)$, or equivalently, if $\varnothing \notin \mathcal{F}$. A filter is called an *ultrafilter* if it is a maximal proper filter for the inclusion relation.

(1) (1pt) Let A be subset of X. Show that the collection $\{B \subseteq X \mid A \subseteq B\}$ is a filter. This is the filter *generated by* A. Such filters are also called *principal*.

(2) (1pt) Let $\mathcal{C}$ be the collection $\{B \subseteq X \mid X - B \text{ is finite}\}$. Show that $\mathcal{C}$ is a filter. This filter is called the *cofinite filter*.

(3) (3pt) Let $\mathcal{F}$ be a proper filter on X. Use Zorn's lemma to prove that there exists an ultrafilter on X that contains $\mathcal{F}$.

(4) (2pt) Let $\mathcal{F}$ be a proper filter on X and suppose that A is a subset of X satisfying the following condition: For all $B \in \mathcal{F}$, the intersection $A \cap B$ is nonempty.
   Show that there exists a proper filter $\mathcal{F}'$ such that $A \in \mathcal{F}'$ and $\mathcal{F} \subseteq \mathcal{F}'$.

(5) (1pt) Let $\mathcal{U}$ be an ultrafilter and A a subset of X. Show that $A \in \mathcal{U}$ or $X - A \in \mathcal{U}$.

(6) (1pt) Show that a principal ultrafilter on X is generated by $\{x\}$ for some $x \in X$.

(7) (1pt) Assume that X is infinite. Show that the cofinite filter $\mathcal{C}$ is proper and not contained in any principal ultrafilter. Conclude that there exist ultrafilters that are not principal.

3. Homework 3 (deadline 3 Dec 2024)

[This exercise is inspired by exercise 44 from the book.] Let X and Y be two partially ordered sets (posets). A function $f\colon X \to Y$ is called:

- *monotone* if for all $x, y \in X$ with $x \le y$ we have $f(x) \le f(y)$;
- *strictly monotone* if for all $x, y \in X$ with $x < y$ we have $f(x) < f(y)$.

(1) (3pts) Assume that X is a linear order and let $f\colon X \to Y$ be a strictly monotone function. Show that for all $x, y \in X$ the following hold:

$$x < y \iff f(x) < f(y) \quad \text{and} \quad x \le y \iff f(x) \le f(y).$$

(2) (4pts) Assume that X is a well-order, and let $f\colon X \to X$ be a *strictly monotone* self-map. Show that for all $x \in X$ we have $x \le f(x)$.

(3) (3pts) Give a counterexample to the preceding exercise where "strictly monotone" is replaced by "monotone".

4. Homework 4 (deadline 10 Dec 2024)

(1) (4pts) Let R be a commutative ring, in other words an L_{rings}-structure satisfying the axioms listed in §2.4.2. Show that R is a local ring if and only if the following L_{rings}-formula is valid:

$$\forall x \exists y (x \cdot y = 1 \vee x \cdot y + 1 = y)$$

This establishes the claim in the book that "being local" is a first-order property of local rings.

(2) (6pts) Let X be a set, and let p be a prime number. Show that there exists a local ring R such that $|X| \le |R|$ and such that

$$p_R \neq 0 \wedge \forall x (p_R \cdot x \neq 1),$$

where $p_R = p \cdot 1_R = 1_R + 1_R + \cdots + 1_R$ (in other words: p viewed as element of R). You may use without proof that the local ring $R = \mathbb{Z}_p$ satisfies these conditions in case X is countable.