CHAPTER 0

Facts

0.1. Profinite sets

Profinite sets are a certain kind of compact Hausdorff spaces with the following properties.

0.1.1. Fact. Let S be a profinite set.

(i) There is *some* way to write $S = \lim_{i \in I} S_i$ in the category CHaus, with the S_i finite and I a cofiltered indexing category. One can arrange that all the transition maps $S_i \to S_j$ are surjective. In that case, the natural maps $S \to S_i$ are also surjective.

Whenever you see someone writing $S = \lim S_i$, there is the standing assumption that: the S_i are finite, the transition maps are surjective, and the indexing category is cofiltered.

(ii) Likewise, if $f \colon S \to S'$ is a surjection of profinite sets, then it can be written as the limit of surjections of finite sets.

(iii) If $S = \lim_{i \in I} S_i$ and M is discrete, then

$$\mathrm{Hom}_{\mathrm{CHaus}}(S, M) = \operatorname*{colim}_{i \in I} \mathrm{Hom}_{\mathrm{CHaus}}(S_i, M)$$

(using that I is cofiltered). Since the transition maps $S_i \to S_j$ are surjective, $\mathrm{Hom}(S_i, M)$ injects naturally into $\mathrm{Hom}(S, M)$ by composition with $S \to S_i$ and the above colimit can viewed as a union.

0.1.2. Fact. (i) For every discrete set S, the Stone–Čech compactification $\beta(S)$ is a profinite set. It is a projective object in CHaus.

(ii) The projective objects in CHaus are also called *extremally disconnected* sets. They are always profinite.

(iii) Every $X \in$ CHaus admits a surjection from an extremally disconnected set, namely $\beta(X^\delta)$, where X^δ denotes the set X with the discrete topology.

0.1.3. Fact. (i) Arbitrary products of profinite sets are profinite.

(ii) Closed subsets of profinite sets are profinite.

(iii) Warning: These two properties typically fail for extremally disconnected sets.

0.2. Ext groups

Let $\mathcal{A}$ be an abelian category with enough projectives.

0.2.1. Fact. For all $X, Y \in \mathcal{A}$ we have $\mathrm{Ext}^0(X, Y) = \mathrm{Hom}(X, Y)$.

0.2.2. Fact. Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}$, and let X be any object. Then there is a long exact sequence of Ext groups:

$$\begin{aligned} 0 \to &\mathrm{Ext}^0(X, A) \to \mathrm{Ext}^0(X, B) \to \mathrm{Ext}^0(X, C) \to \\ &\mathrm{Ext}^1(X, A) \to \mathrm{Ext}^1(X, B) \to \mathrm{Ext}^1(X, C) \to \\ &\mathrm{Ext}^2(X, A) \to \mathrm{Ext}^2(X, B) \to \mathrm{Ext}^2(X, C) \to \dots \end{aligned}$$

0.2.3. Fact. Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}$, and let Y be any object. Then there is a long exact sequence of Ext groups:

$$\begin{aligned} 0 \to &\mathrm{Ext}^0(C, Y) \to \mathrm{Ext}^0(B, Y) \to \mathrm{Ext}^0(A, Y) \to \\ &\mathrm{Ext}^1(C, Y) \to \mathrm{Ext}^1(B, Y) \to \mathrm{Ext}^1(A, Y) \to \\ &\mathrm{Ext}^2(C, Y) \to \mathrm{Ext}^2(B, Y) \to \mathrm{Ext}^2(A, Y) \to \dots \end{aligned}$$

0.2.4. Fact. By definition, an object $C \in \mathcal{A}$ is $\mathrm{Hom}(\_, Y)$-*acyclic* if $\mathrm{Ext}^i(C, Y) = 0$ for all $i > 0$. Such objects can be used to compute $\mathrm{Ext}^i(\_, Y)$: if

$$\dots \to C_2 \to C_1 \to C_0 \to X \to 0$$

is a resolution of X (so an exact complex) and all C_i are $\mathrm{Hom}(\_, Y)$-acyclic, then the groups $\mathrm{Ext}^i(X, Y)$ are the homology groups of the complex

$$\mathrm{Hom}(C_0, Y) \to \mathrm{Hom}(C_1, Y) \to \mathrm{Hom}(C_2, Y) \to \dots$$