CHAPTER 2

Condensed abelian groups and cohomology

2.1. Categorical properties

2.1.1. Definition. A *condensed abelian group* is a group object in the category of condensed sets. Equivalently, it is a functor $A\colon \mathrm{Extr} \to \mathrm{Ab}$ such that

(*i*) $A(\varnothing) = 0$,
(*ii*) $A(S_1 \sqcup S_2) = A(S_1) \times A(S_2)$ for all $S_1, S_2 \in \mathrm{Extr}$.

The category of condensed abelian groups is denoted by Cond(Ab).

2.1.2. Let A be a condensed abelian group. If the underlying condensed set of A is quasiseparated (aka compactological), then A is a compactological abelian group: a compactological space with a compatible structure of abelian group, which means that the operations $(a, b) \mapsto a + b$ and $a \mapsto -a$ are morphisms of compactological spaces.

2.1.3. Theorem. *The category* Cond(Ab) *of condensed abelian groups is an abelian category satisfying the following properties:*

(AB3) all limits exist,*
(AB4) and all products are exact;*
(AB3) all colimits exist,
(AB4) and all coproducts (= direct sums) are exact,
(AB5) and all filtered colimits are exact;
(AB6) and arbitrary products distribute over filtered colimits.

Proof. See Theorem 2.2 in the appendix to Lecture II of [5]. See also Exercise 2.6.1. □

2.1.4. Theorem. *The forgetful functor* $\mathrm{Cond(Ab)} \to \mathrm{Cond(Set)}$ *admits a left adjoint: the "free condensed abelian group" functor*

$$\mathbb{Z}[\_]\colon \mathrm{Cond(Set)} \to \mathrm{Cond(Ab)}$$
$$X \mapsto \mathbb{Z}[X]$$

This means that for any condensed set X and any condensed abelian group A there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Cond(Ab)}}(\mathbb{Z}[X], A) \cong \mathrm{Hom}_{\mathrm{Cond(Set)}}(X, A)$$

that is functorial in X and A. In particular, $\mathbb{Z}[\_]$ preserves all colimits and the forgetful functor preserves all limits.

2.1.5. Let X be a condensed set and $n \in \mathbb{N}$. Then we denote by $\mathbb{Z}[X]_{\leq n}$ the image of the natural map

$$\coprod_{m \leq n} (\{-1,1\} \times X)^m \longrightarrow \mathbb{Z}[X]$$

$$(a_i, x_i)_{1 \leq i \leq m} \longmapsto \sum_i a_i [x_i]$$

2.1.6. Lemma. *Let $f\colon X \to Y$ be a morphism of condensed sets.*

(i) The natural map $\operatorname{colim}_n \mathbb{Z}[X]_{\leq n} \to \mathbb{Z}[X]$ *is an isomorphism.*
(ii) The map f naturally induces a map $\mathbb{Z}[X]_{\leq n} \to \mathbb{Z}]Y]_{\leq n}$.
(iii) If X is compactological, then so are $\mathbb{Z}[X]_{\leq n}$ *and* $\mathbb{Z}[X]$.
(iv) If X is compact Hausdorff, then so is $\mathbb{Z}[X]_{\leq n}$.
(v) If X is finite, then so is $\mathbb{Z}[X]_{\leq n}$.
(vi) If X is profinite, then so is $\mathbb{Z}[X]_{\leq n}$. *Indeed, if* $X = \lim_i X_i$ *with X_i finite, then* $\mathbb{Z}[X]_{\leq n} = \lim_i \mathbb{Z}[X_i]_{\leq n}$.

PROOF. Mostly omitted. The final item is part of [4, Prop 2.1]. See also Exercise 2.6.4. □

2.2. Cohomology

2.2.1. Theorem ([2, §II.5.10])**.** *Let S be a compact Hausdorff space (or more generally a paracompact Hausdorff space). Let A be a discrete abelian group. Then there is a natural isomorphism*

$$H^i_{\text{Čech}}(S, A) = H^i_{\text{sh}}(S, A).$$

2.2.2. Theorem. *Let S be a CW complex (or more generally a cohomologically locally connected space). Let A be a discrete abelian group. Then there is a natural isomorphism*

$$H^i_{\text{sing}}(S, A) = H^i_{\text{sh}}(S, A).$$

PROOF. Assuming that S is paracompact and locally contractible, one can use comparison isomorphisms between singular cohomology, Alexander cohomology and Čech cohomology from [7] together with Theorem 2.2.1 to obtain the comparison isomorphism between singular cohomology and sheaf cohomology. In 2016, Sella proved in [6] that one can drop the paracompactness assumption. In 2021, Petersen gave a short proof in [3] that works under the very mild assumption of cohomological local connectedness: for all $s \in S$ and $k \in \mathbb{Z}$

$$\operatorname*{colim}_{U \ni s} H^k_{\text{sing}}(U, s; A) = 0. \qquad \square$$

2.2.3. DEFINITION. Let S be a condensed set, and let A be a condensed abelian group. We define the *condensed cohomology* of S with coefficients in A to be

$$H^i_{\text{cond}}(S, A) = \operatorname{Ext}^i(\mathbb{Z}[S], A).$$

2.2.4. The next theorem justifies writing $H^i(S, A)$ to denote condensed cohomology $H^i_{\text{cond}}(S, A)$. We now state the remaining main results of this lecture.

2.2.5. Theorem. *Let S be a compact Hausdorff space, and A a* discrete *abelian group. Then condensed cohomology is naturally isomorphic to sheaf cohomology*

$$H^i_{\text{sh}}(S, A) \cong H^i_{\text{cond}}(S, A).$$

2.2.6. Theorem. *Let S be a profinite set, and let A be a complete semi-normed abelian group. Then $H^i(S, A) = 0$ for all $i > 0$.*

2.2.7. Theorem. *Let S be a compact Hausdorff space, and let V be a Banach space. Then $H^i(S, V) = 0$ for all $i > 0$.*

2.2.8. The proof of the last theorem will use the fact that Banach spaces are locally convex. It seems that this use is critical. When we discuss liquid vector spaces, we will have to deal with spaces that are not locally convex, where we cannot apply the above theorem.

2.3. Simplicial intermezzo

2.3.1. We give a quick reminder on Čech nerves and the Čech complex.

Let $f\colon S_0 \to S$ be a surjection of condensed sets. Then we can form the *Čech nerve* of this cover, as follows. Define

$$S_i = \underbrace{S_0 \times_S \cdots \times_S S_0}_{i+1 \text{ copies of } S_0}.$$

Note that there are $i+2$ natural projection maps $\pi_{i,j}\colon S_{i+1} \to S_i$ (so j ranges over $0, \dots, i+1$).

$$\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; S_3 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; S_2 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; S_1 \;\substack{\longrightarrow\\\longrightarrow}\; S_0 \longrightarrow S$$

2.3.2. Lemma. *In the situation above, apply the functor $\mathbb{Z}[\_]$ to obtain*

$$\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathbb{Z}[S_3] \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathbb{Z}[S_2] \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathbb{Z}[S_1] \;\substack{\longrightarrow\\\longrightarrow}\; \mathbb{Z}[S_0] \longrightarrow \mathbb{Z}[S]$$

and take the alternating sum of the maps to obtain a complex

$$\cdots \xrightarrow{d} \mathbb{Z}[S_3] \xrightarrow{d} \mathbb{Z}[S_2] \xrightarrow{d} \mathbb{Z}[S_1] \xrightarrow{d} \mathbb{Z}[S_0] \xrightarrow{d} \mathbb{Z}[S] \longrightarrow 0$$

Then this complex is exact: $\mathbb{Z}[S_\bullet]$ is a resolution of $\mathbb{Z}[S]$.

PROOF. Omitted, but see [Stacks, Tag 01GC]. □

2.3.3. Let A be a condensed abelian group. By applying the functor $\mathrm{Hom}(\_, A)$ and using $\mathrm{Hom}(\mathbb{Z}[X], A) = A(X)$, we get a complex

$$0 \to A(S) \to A(S_0) \to A(S_1) \to A(S_2) \to A(S_3) \to \cdots$$

Note: In general, this complex is *not* exact, since $\mathrm{Hom}(\_, A)$ is not an exact functor.

2.3.4. Now suppose that the map $S_0 \to S$ splits: there is a section $\sigma\colon S \to S_0$. Then we obtain an *extra degeneracy*, for each n we get a map s_{-1}

$$S_n = \overbrace{S_0 \times_S \cdots \times_S S_0}^{n+1 \text{ copies}} \longrightarrow \overbrace{S_0 \times_S \cdots \times_S S_0}^{n+2 \text{ copies}} = S_{n+1}$$
$$(x_0, \dots, x_n) \longmapsto (\sigma(x), x_0, \dots, x_n)$$

where $x \in S$ is the common image of $x_0, \dots, x_n$.

The upshot of such an extra degeneracy is that we get a "contracting homotopy" on our complex above. Write h for the map $\mathbb{Z}[S_n] \to \mathbb{Z}[S_{n+1}]$ induced by s_{-1}. Then we get

$$\cdots \underset{h}{\overset{d}{\rightleftarrows}} \mathbb{Z}[S_3] \underset{h}{\overset{d}{\rightleftarrows}} \mathbb{Z}[S_2] \underset{h}{\overset{d}{\rightleftarrows}} \mathbb{Z}[S_1] \underset{h}{\overset{d}{\rightleftarrows}} \mathbb{Z}[S_0] \underset{h}{\overset{d}{\rightleftarrows}} \mathbb{Z}[S] \longrightarrow 0$$

with the property that $d \circ h + h \circ d = \mathrm{id}$ in each degree (see Exercise 2.6.7).

If A is a condensed abelian group, then the complex

$$0 \to A(S) \to A(S_0) \to A(S_1) \to A(S_2) \to A(S_3) \to \cdots$$

also acquires a contracting homotopy h by functoriality. This means that the complex is exact: if $d(f) = 0$ then $f = d(h(f))$. For future purposes we stress that h is given by pullback along the map s_{-1}.

2.4. Proof of Theorem 2.2.5

2.4.1. Lemma. *Let $S_0 \to S$ be a surjection of profinite sets, and A a discrete abelian group. Form the Čech nerve $S_i = S_0 \times_S \cdots \times_S S_0$. Then the sequence $0 \to A(S) \to A(S_0) \to A(S_1) \to \cdots$ is exact.*

Warning! Even if we choose S_0 to be extremally disconnected, the rest of the S_i are most likely not projective, so we cannot use this sequence to compute $H^i_{\mathrm{cond}}(S, A) = 0$ (yet).

Proof. If S_0 and S are finite, then the map admits a section. Hence the complex is exact by 2.3.4. In general, write $S_0 \to S$ as a cofiltered limit of surjections of finite sets $S_{0,j} \to S_j$. Using the fact that A is discrete, we see that the complex $0 \to A(S) \to A(S_0) \to A(S_1) \to \cdots$ is the filtered colimit of the coresponding complexes for the surjections $S_{0,j} \to S_j$. Indeed, every map $X \to A$ from a profinite set X to a discrete space A must factor via a finite quotient of X. Since a filtered colimit of exact sequences is exact, the result follows. ☐

2.4.2. Lemma. *Let $S_\bullet \to S$ be a Čech nerve, and A a discrete abelian group. Then there is a spectral sequence*

$$H^p(H^q(S_\bullet, A)) \Longrightarrow H^{p+q}(S, A)$$

The terms on the left are formed as follows: first apply the functor $H^q(\_, A)$ to the simplicial object $S_\bullet$. Then form a complex by taking alternating sums of the coface maps and compute the cohomology groups $H^p(\_)$.

Proof sketch. Use Lemma 2.3.2, together with the spectral sequence of a resolution. ☐

2.4.3. Corollary. *Fix a natural number i. If $H^p(H^q(S_\bullet, A)) = 0$ for all $p + q = i$, then $H^i(S, A) = 0$.*

2.4.4. Lemma. *If S is a profinite set and A is a discrete abelian group, then $H^i_{\mathrm{cond}}(S, A) = 0$ for all $i > 0$.*

Proof. Choose a cover of S by an extremally disconnected S_0 and form the Čech nerve as before. We start with two observations. Firstly, note that $H^p(H^0(S_\bullet, A)) = 0$ for all $p > 0$, because $H^0(S_\bullet, A) = A(S_\bullet)$ and Lemma 2.4.1 shows that this has no higher cohomology. Secondly, note that $H^q(S_0, A) = 0$ for all $q > 0$, since S_0 is extremally disconnected. In other words, $H^0(H^q(S_\bullet, A)) = 0$ for all $q > 0$.

Now apply induction on $i > 0$. As induction hypothesis, we may assume that $H^q(S_p, A) = 0$, for all $p + q = i$ with $0 < q < i$. (So the assumption is vacuous if $i = 1$.) In particular, we find $H^p(H^q(S_\bullet, A)) = 0$ under the same conditions. Combined with the two observations and the corollary, we conclude that $H^i(S, A) = 0$ for all profinite sets S. ☐

2.4.5. Lemma. *Let S be a compact Hausdorff space and A a discrete abelian group. Let $s \in S$. Then*

$$\operatorname*{colim}_{U \ni s} H^i_{\mathrm{cond}}(\alpha^* U, A) = \begin{cases} A & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$$

Here the colimit is over the directed system of all open neighborhoods of s in S, and $\alpha^* U$ denotes U, but viewed as a condensed set.

Proof. Since S is locally compact, we can replace the colimit by one over closed neighborhoods $V \ni s$. Pick a surjection $S_0 \to S$ with S_0 profinite and form the Čech nerve S_i. Then for any closed $V \subseteq S$, each $S_i \times_S V$ is profinite and so we can compute $H^i_{\mathrm{cond}}(V, A)$ as the cohomology of the sequence

$$0 \to A(S_0 \times_S V) \to A(S_1 \times_S V) \to \cdots.$$

Using the discreteness of A and compactness of S_0 and S, we can identify the colimit of these sequences with the sequence

$$0 \to A(S_0 \times_S \{s\}) \to A(S_1 \times_S \{s\}) \to \cdots.$$

(Specifically, any function $f : S_i \times_S V \to A$ becomes constant in the second argument after shrinking V.) The latter sequence computes $H^i_{\mathrm{cond}}(\{s\}, A)$, which is equal to A if $i = 0$ and vanishes otherwise. □

Proof of Theorem 2.2.5. Once again, we resort to a spectral sequence argument. Let $\mathcal{H}^q$ denote the sheaf on the topological space S obtained as the sheafification of

$$U \mapsto H^q_{\mathrm{cond}}(\alpha^* U, A).$$

We claim without proof the Leray spectral sequence:

$$H^p_{\mathrm{sh}}(S, \mathcal{H}^q) \Longrightarrow H^{p+q}_{\mathrm{cond}}(S, A).$$

Hence we are done if we show that $\mathcal{H}^0 = A$ and $\mathcal{H}^i = 0$ for $i > 0$. These are conditions that can be checked on stalks, and hence we are done by the preceding lemma. □

2.5. Proof of Theorem 2.2.6 and Theorem 2.2.7

2.5.1. Definition. Let $K > 0$ be a real number. A complex $A \to B \to C$ of semi-normed abelian groups is *K-bounded exact* at B if for all $b \in B$ and $\varepsilon > 0$ there exists an $a \in A$ such that

$$\|b - da\| \le K\|db\| + \varepsilon.$$

2.5.2. Lemma. *Let $K > 0$ be a real number, and let $A \to B \to C \to D$ be a complex of complete normed abelian groups that is K-bounded exact at B and C. Then the complex is exact at C.*

Proof. Define two decreasing sequences of positive real numbers, via $\delta_i = 2^{-(i+1)}$ and $\varepsilon_i = \delta_i/(2K)$. Fix $y \in C$ with $d(y) = 0$. By assumption we can find $w_i \in B$ such that

$$\|y - dw_i\| \le K\|dy\| + \varepsilon_i = \varepsilon_i.$$

Now there is no reason to expect the w_i to converge. However, we can compute

$$\begin{aligned} \|d(w_{i+1} - w_i)\| &= \|(dw_{i+1} - y) + (y - dw_i)\| \\ &\le \|dw_{i+1} - y\| + \|y - dw_i\| \qquad (*) \\ &\le \varepsilon_{i+1} + \varepsilon_i \le 2\varepsilon_i \end{aligned}$$

This suggests a telescoping procedure to correct the errors. Once again, by assumption we can find $z_i \in A$ such that

$$\|(w_{i+1} - w_i) - dz_i\| \leq K\|d(w_{i+1} - w_i)\| + \delta_i.$$

Now define

$$x_i = w_i - \sum_{j<i} dz_j.$$

The following computation shows that $(x_i)_i$ is a Cauchy sequence.

$$\begin{aligned}\|x_{i+1} - x_i\| &= \|w_{i+1} - w_i + \sum_{j<i} dz_j - \sum_{j<i+1} dz_j\| \\ &= \|w_{i+1} - w_i - dz_i\| \\ &\leq K\|d(w_{i+1} - w_i)\| + \delta_i \qquad \text{by the assumption on } z_i \\ &\leq 2K\varepsilon_i + \delta_i \qquad \text{by } (*) \\ &\leq 2^{-i}\end{aligned}$$

Let x denote the limit of the sequence $(x_i)_i$. Since $d^2 = 0$, note that $dx_i = d(w_i)$. Hence $\|y - dx_i\| = \|y - dw_i\| \leq \varepsilon_i$ and we conclude that $y = dx$. □

2.5.3. Lemma. *Let A be a semi-normed abelian group, and let $S_\bullet \to S$ be the Čech nerve of a surjection of finite sets. Then the Čech complex*

$$0 \to C(S, A) \to C(S_0, A) \to C(S_1, A) \to \cdots$$

is 1-bounded exact.

Proof. Since S and all the S_i are finite, we get a contracting homotopy (§2.3.4) of the form

$$h(f) = f \circ s_{-1}, \qquad f \in C(S_{i+1}, A)$$

Hence $\|h(f)\| = \sup_x \|f(s_{-1}(x))\| \leq \|f\|$, which means that h has operator norm ≤ 1. Using the identity $dh + hd = \mathrm{id}$, we conclude that for all f we have

$$\|f - dh(f)\| = \|h(df)\| \leq \|d(f)\|.$$

This gives the result. □

2.5.4. Lemma. *Let $K > 0$ be a real number, and let*

$$A_0 \to A_1 \to A_2 \to \dots$$

be a complex of semi-normed abelian groups. Assume that the underlying complex of condensed abelian groups is a filtered colimit of complexes of semi-normed abelian groups

$$A'_{i,0} \to A'_{i,1} \to A'_{i,2} \to \dots$$

such that all the inclusion maps $A'_{i,\bullet} \hookrightarrow A_\bullet$ are isometries. If the complexes $A'_{i,\bullet}$ are K-bounded exact, then so is the complex $A_\bullet$.

Proof. Pick $y \in A_{n+1}$ and $\varepsilon > 0$. Then there exists a j and $\tilde{y} \in A'_{j,n+1}$ that maps to y. By assumption there is a $\tilde{x} \in A'_{j,n}$ with

$$\|\tilde{y} - d\tilde{x}\| \leq K\|d\tilde{y}\| + \varepsilon.$$

Take $x \in A_n$. Since the inclusions $A'_{j,\bullet} \hookrightarrow A_\bullet$ are isometries, the result follows:

$$\|y - dx\| \leq K\|dy\| + \varepsilon. \qquad \square$$

2.5.5. Lemma. *Let $K > 0$ be a real number, and let $A \to B \to C$ be a complex of semi-normed abelian groups. Then $A \to B \to C$ is K-bounded exact if and only if the complex of completed semi-normed abelian groups $\hat{A} \to \hat{B} \to \hat{C}$ is K-bounded exact.*

Proof. Exercise 2.6.8. $\square$

2.5.6. Lemma. *Let A be a semi-normed abelian group, and let $S_\bullet \to S$ be the Čech nerve of a surjection of profinite sets. Then the Čech complex*

$$0 \to C(S, A) \to C(S_0, A) \to C(S_1, A) \to \cdots$$

is 1*-bounded exact.*

Proof. A crucial ingredient in this proof is the following observation: if X is a profinite set, then the group $C(X, A)$ of continuous functions is the completion of the group $C(X, A^\delta)$ of locally constant functions with respect to the sup-norm.

Write $S_0 \to S$ as a cofiltered limit of surjections of finite sets. Then the Čech complex

$$0 \to C(S, A) \to C(S_0, A) \to C(S_1, A) \to \cdots$$

is the completion of the filtered colimit of the Čech complexes of the surjections between finite sets. The inclusion maps from the finite stages into the $C(S_\bullet, A)$ are isometries. Hence the result follows by combining the preceding three lemmas. $\square$

Proof of Theorem 2.2.7. Let S be a compact Hausdorff space and V a Banach space. Let $S_0 \to S$ be a cover by a profinite set, and form the Čech nerve $S_\bullet$. We will now show that $H^i(S, A) = 0$ for $i > 0$. Since the claim is true for profinite sets, it suffices to show that

$$0 \to C(S, V) \to C(S_0, V) \to C(S_1, V) \to \cdots$$

is exact. We may as well show that it is 1-bounded exact (by Lemma 2.5.2).

Fix $f \in C(S_i, V)$ and $\varepsilon > 0$. Let π_i denote the canonical map $S_i \to S$. For each point $s \in S$, consider the restriction $f_s = f|_{\pi_i^{-1}(s)}$ to the fiber above s. By Lemma 2.5.6 applied to the profinite set $\pi_i^{-1}(s)$ and the function f_s, we know that there exists a $g_s \in C(\pi_{i-1}^{-1}(s), V)$ such that

$$\|f_s - dg_s\| \leq \|df_s\| + \varepsilon/2 \leq \|df\| + \varepsilon/2.$$

By Dugundji's extension theorem [1] (a generalization of Tietze's extension theorem) there is a $\tilde{g}_s \in C(S_{i-1}, V)$ that extends g_s. Let $U_s \subseteq S$ be a neighborhood of s such that

$$\|(f - d\tilde{g}_s)|_{\pi^{-1}(U)}\| < \|df\| + \varepsilon.$$

By compactness, finitely many U_s cover S. Pick such a finite collection of $(U_s, \tilde{g}_s)$ and relabel them $(U_1, g_1), \dots (U_n, g_n)$. Now choose a partition of unity $1 = \sum_{j=1}^n \rho_j$ by functions $\rho_j \in C(S, \mathbb{R}_{\geq 0})$ such that ρ_j has support in U_j. We view the ρ_j as functions on the S_i be composition with π_i.

Put $g = \sum_j \rho_j g_j$ and observe that

$$\begin{aligned} \|f - dg\| &= \|f - \sum_j d(\rho_j g_j)\| \\ &= \|f - \sum_j \rho_j dg_j\| && \text{since } \rho_j \text{ are constant on fibres of } \pi_i \\ &= \|\sum_j \rho_j (f - dg_j)\| \\ &\le \sum_j \rho_j \|(f - dg_j)\|_{U_j} && \text{using local convexity} \\ &\le \|df\| + \varepsilon. \end{aligned}$$

□

2.6. Exercises

2.6.1. Exercise. In this exercise, we prove Theorem 2.1.3, by considering condensed abelian groups as functors

$$\mathrm{Extr}^{\mathrm{op}} \to \mathrm{Ab}$$

that send finite disjoint unions in Extr to finite products in Ab.

Let $i \mapsto M_i \colon I \to \mathrm{Cond(Ab)}$ be a diagram. Show that the limit and colimit of this diagram are given by $S \mapsto \lim_i M_i(S)$ and $S \mapsto \mathrm{colim}_i\, M_i(S)$ respectively. Now deduce the properties of Cond(Ab) from the corresponding properties of Ab. (Hint: use that in Ab finite products are the same as finite coproducts, i.e., they are biproducts. Hence formation of arbitrary (co)limits commutes with finite biproducts.)

2.6.2. Exercise. Show that the subcategory of Cond(Ab) on all compactological abelian groups is stable under all limits and filtered colimits whose transition maps are monomorphisms.

2.6.3. Exercise. Let S be an extremally disconnected set. Show that $\mathbb{Z}[S]$ is a projective object in Cond(Ab).

2.6.4. Exercise. Prove parts of Lemma 2.1.6, namely:

(*i*) The map f naturally induces a map $\mathbb{Z}[X]_{\le n} \to \mathbb{Z}]Y]_{\le n}$.
(*ii*) If X is finite, then so is $\mathbb{Z}[X]_{\le n}$.
(*iii*) If X is profinite, then so is $\mathbb{Z}[X]_{\le n}$. Indeed, if $X = \lim_i X_i$ with X_i finite, then $\mathbb{Z}[X]_{\le n} = \lim_i \mathbb{Z}[X_i]_{\le n}$.

2.6.5. Exercise. Let A be a condensed abelian group. Show that A is quasiseparated if and only if $0 \to A$ is closed.

2.6.6. Exercise. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of condensed abelian groups. If A and C are quasiseparated (aka compactological) show that B is also quasiseparated. (Hint: use the preceding exercise.)

2.6.7. Exercise. In the context of 2.3.4 prove the identity $d \circ h + h \circ d = \mathrm{id}$ in each degree.

2.6.8. Exercise. Let $K > 0$ be a real number, and let $A \to B \to C$ be a complex of semi-normed abelian groups. Show that $A \to B \to C$ is K-bounded exact if and only if the complex of completed semi-normed abelian groups $\hat{A} \to \hat{B} \to \hat{C}$ is K-bounded exact.

2.6.9. Exercise. Let $f\colon V \to W$ be a surjective continuous linear map between two Banach spaces. Show that f is an epimorphism when considered as morphism of condensed sets. (Hint: for profinite sets S show that that the induced map on locally constant functions $C(S, V^\delta) \to C(S, W^\delta)$ is surjective. Now use the Banach open mapping theorem, to deduce that $C(S, V^\delta) \to C(S, W^\delta) \to 0$ is K-bounded exact for some K. Conclude by passing to the completion.)

2.6.10. Exercise. Let $f\colon V \to W$ be a surjective continuous linear map between two Banach spaces, and let K be a compact Hausdorff space. Show that the induced map $C(K, V) \to C(K, W)$ is surjective. (Hint: Use the preceding exercise and Theorem 2.2.7.)

2.6.11. Exercise. Prove the Bartle–Graves theorem: every surjective continuous linear morphism of Banach spaces admits a continuous section. (Hint: the section does not have to be linear! Use the preceding exercise and property (AB5).)