CHAPTER 1

Condensed sets

1.1. Motivation: mixing algebra and topology

Both in analysis and in algebra, we encounter algebraic structures—abelian groups, say—that are also of a topological nature.

- $\mathbb{R}$, $\mathbb{C}$. Continuous ($\mathbb{R}$- or $\mathbb{C}$-valued) functions (on a compact Hausdorff space, say). L^p functions, smooth functions, ...
- Power series rings $\mathbb{Z}[[t]]$, with the topology of t-adic convergence. $\mathbb{Z}_p$, $\mathbb{Q}_p$.
- Ordinary abelian groups, regarded as having the discrete topology.

(You might like to consider what distinguishes these three classes of examples. There are several possible answers.)

Each of these examples has an important *completeness* property, that allows us to construct elements by giving a sequence of better and better approximations. In order to formulate this property, we need to keep track of (at least) the topology of our abelian group.

The objective of condensed mathematics is to systematically extend algebraic methods that work very well for ordinary abelian groups (such as homological algebra) to a setting that retains this kind of "topological" information.

An obvious thing to try, if interested in say abelian groups that also have a topological structure, is to work in the category of topological abelian groups. A basic tool that works well for *ordinary* abelian groups is the notion of short exact sequence: a complex

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

such that $A = \ker g$, and $\operatorname{coker} f = C$. In particular, we expect to be able to complete any injective $f : A \to B$ or surjective $g : B \to C$ to a short exact sequence. In topological abelian groups, this works some of the time. For example, there is a short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$$

because the circle $\mathbb{R}/\mathbb{Z}$ has the quotient topology from $\mathbb{R}$, and $\mathbb{Z} \subseteq \mathbb{R}$ has the subspace topology.

But what if we define

$$\mathbb{R}^\delta := \text{real numbers, with the discrete topology}$$

and look for a short exact sequence

$$0 \to \mathbb{R}^\delta \xrightarrow{i} \mathbb{R} \to C \to 0 \qquad (?)$$

The cokernel C should satisfy, for any topological abelian group D,

$$\begin{aligned}\operatorname{Hom}(C, D) &= \{\, \text{continuous group homomorphisms } \varphi : \mathbb{R} \to D \mid \varphi|_{\mathbb{R}^\delta} = 0 \,\} \\ &= 0\end{aligned}$$

and so C is just the zero group. But then $\ker(\mathbb{R} \to 0)$ is $\mathbb{R}$, and not $\mathbb{R}^\delta$. So we were unable to form a short exact sequence.

From this example, we learn the following:

1. Since we want a theory that has
   (a) the topological abelian group $\mathbb{R}$,
   (b) the discrete abelian group $\mathbb{R}^\delta$,
   (c) a good theory of short exact sequences,

   we *must also have* some mysterious object $\mathbb{R}/\mathbb{R}^\delta \neq 0$, whether or not we have any other interest in this object.
2. The same problem would arise in any other category of "sets with structure", e.g., topological metric spaces (assuming that $\mathbb{R}$ and $\mathbb{R}^\delta$ have the same underlying set).
3. This problem arises because the map $f : \mathbb{R}^\delta \to \mathbb{R}$ is injective and surjective, but not a homeomorphism (isomorphism).

1.1.1. Observation. The category of *compact Hausdorff spaces* does not have this deficiency! If $f : X \to Y$ is a continuous map of compact Hausdorff spaces that is injective and surjective, then f is a homeomorphism. Furthermore, compact Hausdorff spaces have "topological structure".

Unfortunately, nearly all of our motivating examples fail to be compact. Notably, even $\mathbb{Z}$, the free abelian group on one generator, is not compact. As it turns out, this is the only "problem" with the category CHaus of compact Hausdorff spaces that we need to "correct" in order to make it a suitable setting for our purposes. This suggests the following strategy, which is also a brief outline of this course.

1. Expand CHaus to a larger category in a way that retains its existing good properties, while also including (e.g.) infinite discrete sets.
2. Study abelian group objects in this category, using methods from homological algebra.
3. Formulate "completeness conditions" on these abelian group objects.

The category we will construct in step 1 is the *category of condensed sets*, which is the subject of this lecture. We will describe this category from two perspectives.

- It is obtained from CHaus by a formal "cocompletion" process, so it behaves like an enlarged mathematical universe in which the basic building blocks are the compact Hausdorff spaces. This perspective is good for setting up the general theory, but it doesn't give a very good idea of what an individual condensed set "looks like".
- There is a class of condensed sets called "quasiseparated" which can equivalently be described as sets with certain extra structure, similar to but not exactly the same as a topology. All the examples mentioned at the start of this lecture belong to this class. This perspective is useful for geometric intuition and for doing computations with specific condensed sets.

1.2. Condensed sets as a cocompletion

1.2.1. Before giving an actual definition of condensed sets, here is a list of desiderata that we would want from the category Cond of condensed sets.

(0) It should be a similar as possible to the category of sets: It should have colimits and limits, and they should interact as expected.
(1) For each $K \in \mathrm{CHaus}$, there should be a $\underline{K} \in \mathrm{Cond}$ and $\mathrm{Cond}(K, K') = \mathrm{CHaus}(K, K')$.
(2) In CHaus we have finite disjoint unions. The construction $K \mapsto \underline{K}$ should preserve those.
(3) If $E \rightrightarrows K$ is an equivalence relation in CHaus (i.e., $E \subseteq K \times K$ is a closed equivalence relation) then $\underline{K/E} = \underline{K}/\underline{E}$.
Equivalently, if $q\colon K \to L$ is surjective (we could take $E = K \times_L K$) then
$$\underline{K \times_L K} \rightrightarrows \underline{K} \to \underline{L}$$
is a coequalizer.
(4) Every condensed set should be glued from compact Hausdorff spaces.

1.2.2. For any $X \in \mathrm{Cond}$, we glean that X is determined by all of sets
$$\mathrm{Cond}(\underline{K}, X) =: X(K)$$
of morphisms from compact Hausdorff spaces K into X. Whenever $f\colon K \to L$ is a morphism in CHaus, we get a map
$$X(L) \to X(K)$$
that is "precomposition with f".

1.2.3. Definition (preliminary). A *"condensed set"* is a functor
$$X\colon \mathrm{CHaus}^{\mathrm{op}} \to \mathrm{Set}$$
such that

- $X(K_1 \sqcup \cdots \sqcup K_n) = X(K_1) \times \cdots \times X(K_n)$
- If $q\colon K \to L$ is surjective, then
$$X(L) \to X(K) \rightrightarrows X(K \times_L K)$$
is an equalizer.

1.2.4. Warning. There is a size issue, because CHaus is a large category. We will indicate how to resolve this problem, so that we can turn the preliminary definition into an actual definition. Let κ be a "big" cardinal. A precise meaning of a "big" cardinal could be *strong limit cardinal.* But the cardinality of $\mathcal{P}(\mathcal{P}(\mathbb{R}))$ is good enough for most purposes. In the end, we will take a "colimit" over all κ, and therefore the precise choice does not matter so much.

1.2.5. Definition. A *condensed set* is a functor
$$X\colon \mathrm{CHaus}^{\mathrm{op}} \to \mathrm{Set}$$
such that

- $X(K_1 \sqcup \cdots \sqcup K_n) = X(K_1) \times \cdots \times X(K_n)$

- If $q: K \to L$ is surjective, then
$$X(L) \to X(K) \rightrightarrows X(K \times_L K)$$
is an equalizer

and such that X is a colimit of a *small* diagram of K_α, or equivalently $\coprod K_\alpha \twoheadrightarrow X$.

1.2.6. If Y is a topological space, then we can attach a "condensed set" to it

$$\begin{aligned} \underline{Y}: \mathrm{CHaus}^{\mathrm{op}} &\to \mathrm{Set} \\ K &\mapsto C(K, Y) \end{aligned}$$

To ensure that $\underline{Y}$ is an actual condensed set, we should assume that Y is T1. Indeed, the Sierpinski space $\mathcal{S}$ consisting of two points, one open and one closed, does not give rise to a condensed set: maps $K \to S$ correspond to open subset of K; hence there is no possibility of bounding $\mathcal{S}(K)$. In particular, $\underline{\mathcal{S}}$ is not the colimit of a *small* diagram of compact Hausdorff spaces.

1.2.7. Definition. Y is *(κ-)compactly generated* if its topology is generated by continuous maps from (κ-)small compact Hausdorff spaces. In other words, a map $\varphi: Y \to Z$ is continuous if and only if $\varphi \circ g$ is continuous for all continuous $g: K \to Y$ from (κ-)small $K \in \mathrm{CHaus}$.

1.2.8. Example. If Y is sequential, then its topology is determined by continuous maps from $\mathbb{N} \cup \{\infty\}$. In particular, this happens if Y is a metric space.

1.2.9. Fact. The construction $Y \to \underline{Y}$ is fully faithful on κ-cgTop. See [1, Prop 1.7].

1.2.10. Recall that $E \in \mathrm{CHaus}$ is *extremally disconnected* if and only if it is projective in CHaus. If S is discrete, then the Stone–Čech compactification βS is extremally disconnected.

1.2.11. Theorem. *Let $f: X \to Y$ be a morphism in* Cond. *The following are equivalent:*

(i) For any $Z \in$ Cond, *the induced map*
$$\mathrm{Cond}(Y, Z) \to \mathrm{Cond}(X, Z)$$
is injective. I.e., f is an epimorphism.

(ii) The diagram
$$X \times_Y X \rightrightarrows X \to Y$$
is a coequalizer. I.e., f is an effective epimorphism.

(iii) For any $g: K \to Y$, with $K \in$ CHaus, *there exists a finite family $q_i: K_i \to K$ of jointly surjective maps and lifts $l_i: K_i \to X$ such that $f \circ l_i = g \circ q_i$.*

$$\begin{array}{ccccc} & & & & X \\ & & \overset{\exists \sqcup l_i}{\nearrow} & & \downarrow f \\ K_1 \sqcup \cdots \sqcup K_n & \xrightarrow[\sqcup q_i]{} & K & \xrightarrow[g]{} & Y \end{array}$$

(iv) $f(E): X(E) \to Y(E)$ is surjective for all extremally disconnected E.

(v) $f(\beta S): X(\beta S) \to Y(\beta S)$ is surjective for all discrete sets S.

Proof. Omitted. □

1.2.12. Definition. A morphism $f: X \to Y$ of condensed sets is called *surjective* if it satisfies the equivalent conditions of Theorem 1.2.11. We also say that f is a *cover*

1.2.13. Example. Consider for f the identity function $\mathbb{R}^\delta \to \mathbb{R}$, and for g the inclusion $[0,1] \hookrightarrow \mathbb{R}$. Then we cannot find $K_1 \sqcup \cdots \sqcup K_n$ that cover $[0,1]$ in such a way that we get a lift. After all, the image of a compact set in $\mathbb{R}^\delta$ must be finite. Hence we see that $\mathbb{R}^\delta \to \mathbb{R}$ is not *surjective*.

1.2.14. Corollary (to Theorem 1.2.11)**.** *The category* Cond *of condensed sets has enough projectives.*

Proof. Let X be a condensed set. Then there exists a collection of compact Hausdorff spaces K_α and a cover $\bigsqcup_\alpha K_\alpha \twoheadrightarrow X$. Then

$$\bigsqcup_\alpha \beta(K_\alpha^\delta) \twoheadrightarrow \bigsqcup_\alpha K_\alpha \twoheadrightarrow X$$

is a cover by a projective object, where K_α^δ is the discrete space with the same underlying set as K_α. □

1.3. Quasiseparated condensed sets

1.3.1. Definition. A monomorphism $f\colon X \to Y$ in Cond is *closed* if for every $K \to Y$ the fiber product $K \times_Y X$ is a compact Hausdorff space.

1.3.2. Example. The identity function $\mathbb{R}^\delta \to \mathbb{R}$ is not closed.

1.3.3. Definition. A condensed set X is *separated* if $\Delta\colon X \to X \times X$ is closed.

1.3.4. Example. If Y is a topological space, then $\underline{Y}$ is separated if and only if Y is compactly generated weak Hausdorff (CGWH).

1.3.5. Theorem. *There is another categorical notion "quasiseparated" objects. If X is a condensed set, then X is quasiseparated if and only if X is separated.*

Proof. Exercise 1.5.8. □

1.3.6. Let X be a quasiseparated condensed set. Consider a map $K \to X$, and form the pullback square $E = K \times_X K$. Note that it is the same as the pullback of the diagonal $\Delta\colon X \to X \times X$ to $K \times K$. Hence K/E is compact Hausdorff, and $K/E \to X$ is injective. We conclude that X can be covered by compact Hausdorff subobjects.

1.4. Compactological spaces

1.4.1. It turns out that the notion of quasiseparated condensed sets is equivalent to so-called *compactological spaces*. We will now review this "ancient" but little-known theory introduced by Waelbroeck in chapter III of [2].

1.4.2. Definition. A *bornology* on a set X consists of a set $\mathcal{B}$ of subsets of X: the *small* subsets of X. It must satisfy the following axioms:

- every finite subset of X is small;
- every subset of a small subset is small;
- finite unions of small subsets are small.

A function $X \to Y$ between sets endowed with a bornology is called *small* if the image of every small subset of X is small in Y.

1.4.3. Example. Let X be a metric space. Then the collection of sets $S \subseteq X$ that are contained in some closed ball $B_r(x) \subseteq X$ form a bornology on X.

1.4.4. Example. On every set X, one can form the bornology of finite subsets of X.

1.4.5. DEFINITION. Let X be a set. A *compactology* on X consists of a topology and a bornology on X that satisfy the following compatibility conditions:

- The topology is coherent with the subspace topologies on the closed small sets: a subset $S \subseteq X$ is closed (resp. open) in X if and only if for every closed small set $B \subseteq X$ the intersection $S \cap B \subseteq B$ is closed (resp. open) in B.
- Every small set is contained in a set that is small *and* compact Hausdorff.

A *compactological space* is a set X endowed with a distinguished compactology.

A morphism of compactological spaces $X \to Y$ is a function that is continuous and small.

1.4.6. Lemma. *Let X be a compactological space. The closure of a small set is compact Hausdorff. In particular, a closed and small subset of X is compact Hausdorff. The converse is not always true: there may exist compact Hausdorff subsets that are not small.*

PROOF. Let B be a small set. Then it is contained in a small compact Hausdorff set $K \subseteq X$. Hence the closure $\bar{B} \subseteq X$ is a closed subset of K, therefore compact Hausdorff. In particular, a closed small subset of X is compact Hausdorff. In the following example we exhibit a compact Hausdorff compactological space that is not small. □

1.4.7. Example. Let X be an uncountable compact metric space, for example $X = [0, 1]$. Endow X with the bornology given by subsets of countable closed subsets of X. Then X is a compactological space, and X is compact Hausdorff, but not small.

Indeed, X is metrizable, hence sequential. In other words, the topology is completely determined by the convergent sequences in X. Since a convergent sequence is a countable closed subset of X this means that the topology is coherent with the closed small sets. See also Exercise 1.5.1.

1.4.8. Lemma. *Let X be a compactological space. Then the topology on X makes X into a compactly generated weakly Hausdorff space.*

PROOF. To show that X is compactly generated we need to show that the topology is coherent with the family of all compact subsets of X. Suppose that S is a subset of X such that $S \cap K$ is closed in K for every compact $K \subseteq X$. Then certainly $S \cap B$ is closed in B for every closed small $B \subseteq X$, since every such B is compact by Lemma 1.4.6. In other words, such a set S is closed in X. Conversely, suppose that S is a closed subset of X, and let K be a compact subset of X. Then $S \cap K$ is closed in K by definition.

To show that X is weakly Hausdorff, we need to show that every continuous map $f\colon K \to X$ from a compact Hausdorff K has closed image. Let $B \subseteq X$ be closed and small. Then $f(K) \cap B$ is closed in B since it is compact, being the image of the set $f^{-1}(B)$ which is compact in K since it is closed. □

1.4.9. Every CGWH space can be viewed as a compactological space by declaring every subset of a compact Hausdorff subset to be small. In particular, whenever we talk of a compact Hausdorff space, we view it as a compactological space in this way: all its subsets are small.

Example 1.4.7 shows that not all compactological spaces arise in this manner.

1.4.10. Warning. In general, the "ordinary" product topology on the product of two CGWH spaces is not itself CGWH. However the category of CGWH spaces does have products (in fact it has all limits). One should add open and closed subsets in order to make the topology coherent with the family of compact subsets and obtain the CGWH product topology. If X is a CGWH space, then the diagonal in $X \times X$ is closed for the CGWH product topology, but not necessarily for the "ordinary" product topology. This explains why X need not be Hausdorff, even though it is the union of compact Hausdorff spaces.[1]

This process of refining the topology to make it compactly generated goes by the name of k-ification. We will outline the process below, since the same remarks apply to compactological spaces.

1.4.11. Definition. Let X be a topological space. Define a new topology on X be declaring $S \subseteq X$ to be closed if and only if $S \cap K$ is closed for all compact subset K in the original topology. It is easy to see that these closed sets indeed form a topology. Denote this new topological space by X^{cg}. It is called the *k-ification* of X.

1.4.12. Fact.

- (i) If X is a topological space, then X^{cg} has the same compact subsets as X.
- (ii) Let X and Y be topological spaces, and assume that Y is compactly generated. Then every continuous function $X \to Y$ factors uniquely via $X \to X^{\mathrm{cg}}$.

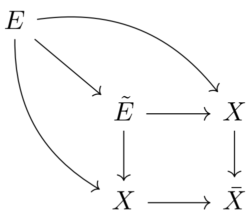
1.4.13. Corollary. *The construction $X \mapsto X^{cg}$ is functorial. It exhibits compactly generated spaces as a reflective subcategory of* Top.

1.4.14. Warning. If X is a compactological space, and $\iota\colon S \hookrightarrow X$ is an injective morphism, then the isomorphism class of S is not determined by the subset of X that is the image of ι. This should not come as a surprise, since the same phenomenon occurs in topology: take the identity function between the discrete space X^{δ} and any non-discrete space X.

If $S \subseteq X$ is a closed subset, then there is a natural induced compactology on S. In general, we warn against identifying a subobject (aka an injective morphism) with the underlying set of its image.

1.4.15. The category of compactological spaces is a rather nice category, having all limits and colimits. But it shares a defect with categories of topological spaces: quotients are rather crude, and only quotients by closed equivalence relations are well-behaved.

If X is a compactological space, and $E \hookrightarrow X \times X$ is an equivalence relation, then we can form the quotient map $q\colon X \to \bar{X} = X/E$. We can also form a new equivalence relation $\tilde{E} \hookrightarrow X \times X$ as the pullback $X \times_{\bar{X}} X$.



[1]See `https://mathoverflow.net/questions/47702`, *Why the "W" in CGWH (compactly generated weakly Hausdorff spaces)?*, and `https://mathoverflow.net/questions/204167` *Why should have Peter May worked with CGWH instead of CGH in "The Geometry of Iterated Loop Space"?*, for some more details and historical remarks.

Although there is a natural injection $E \hookrightarrow \tilde{E}$, this is typically not an isomorphism: $\tilde{E}$ is the closure of E with the bornology induced from $X \times X$. The category of condensed sets solves this defect, by adding formal quotients X/E for non-closed equivalence relations $E \hookrightarrow X \times X$.

1.5. Exercises

1.5.1. Exercise.

(i) (This exercise shows that metric spaces are sequential.) Let $f\colon X \to Y$ be a function between metric spaces, and let $x \in X$ be a point. Prove that f is continuous at x if and only if the following condition holds: for all sequences $\alpha\colon \mathbb{N} \to X$ that converge to x the image sequence $f \circ \alpha$ converges to $f(x)$.

(ii) Deduce that $Y \mapsto \underline{Y}$ is a fully faithful functor from metric spaces to condensed sets.

(iii) Assume that X has the Heine–Borel property: a subset of X is closed and bounded if and only if it is compact. Conclude that there is a natural compactology on X in which the small sets are precisely the bounded subsets of X.

1.5.2. Exercise. Let S be a set endowed with a bornology.

(i) Show that there exists a compactological space $\gamma(S)$ and a small function $S \to \gamma(S)$ with the following universal property: every small function $S \to X$ to a compactological space X extends uniquely to a morphism of compactological spaces $\gamma(S) \to X$.

(ii) If every subset of S is small, then $\gamma(S) \cong \beta(S)$.

1.5.3. Exercise. A morphism of compactological spaces is *strongly surjective* if every small subset of the codomain is the image of a small subset in the domain.

(i) Observe that a strongly surjective morphism is surjective.

(ii) Show that a morphism of compactological spaces is strongly surjective if and only if it is an surjection of condensed sets.

1.5.4. Exercise. Let X be a compact Hausdorff space, and view X as a compactological space by endowing it with its natural bornology (every set is small). Let $X^{(\omega)}$ denote the compactological space whose underlying topological space is X, but endowed with the bornology in which countable closed sets are small. Observe that

(i) the identity function $X^{(\omega)} \to X$ is a morphism;

(ii) if X is uncountable, this morphism is not strongly surjective;

(iii) if X is uncountable, then $X^{(\omega)}$ is not quasicompact (in the categorical sense; see below).

1.5.5. Exercise. Show that the essential image of the natural functor

$$\mathrm{Cond(Set)} \to [\mathrm{Extr}^{\mathrm{op}}, \mathrm{Set}]$$

consists of the full subcategory on those functors that send finite disjoint unions of extremally disconnected sets to finite products. In other words, the functors that satisfy the following two conditions:

(i) $X(\varnothing) = *$.

(ii) The natural map $X(S_1 \sqcup S_2) \to X(S_1) \times X(S_2)$ is an isomorphism for any two extremally disconnected sets S_1 and S_2.

1.5.6. Exercise.

(i) Show that an arbitrary coproduct of projective objects is projective.

(*ii*) Show that a retract of projective objects is projective.
(*iii*) Conclude that every projective condensed set is a retract of a coproduct of spaces of the form $\beta(S)$.

1.5.7. Exercise. Let S be a set that contains at least two elements. Show that the evaluation functor at $\beta(S)$

$$\begin{aligned} \mathrm{Cond}(\mathrm{Set}) &\longrightarrow \mathrm{Set} \\ X &\longmapsto X(\beta(S)) \end{aligned}$$

does not preserve binary coproducts.

1.5.8. Exercise. Here is the definition of quasiseparated (and quasicompact) from SGA4, specialized to condensed sets.

- A condensed set X is *quasicompact* if for any cover $\coprod Y_\alpha \twoheadrightarrow X$, some finite collection of the Y_α already cover X.
- A condensed set X is *quasiseparated* if for any diagram

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

with Y and Z quasicompact, the fiber product $Y \times_X Z$ is also quasicompact.

(*i*) Show that a condensed set X is quasicompact if and only if it can be covered by a single compact Hausdorff space.
(*ii*) Show that a condensed set X is quasicompact and quasiseparated if and only if it is a compact Hausdorff space.
(*iii*) Show that X is quasiseparated if and only if it is separated (the diagonal is closed). Hint for the proof: First prove that if K is a compact Hausdorff space and W is a quasicompact subobject of K, then W is also a compact Hausdorff space.

1.5.9. Exercise. Show that a condensed subobject of a compactological space is quasiseparated (aka a compactological space).

1.5.10. Exercise. This exercise constructs the "Hausdorffification" of a condensed set. In other words: we show that compactological spaces form a reflective subcategory of condensed sets. Let X be a condensed set.

(*i*) Show that there is a smallest closed equivalence relation $\bar{\Delta} \subseteq X \times X$. (You will need to use/prove that an arbitrary intersection of closed subobjects is closed.)
(*ii*) Show that the quotient X^{qs} of X by the equivalence relation $\bar{\Delta}$ is quasiseparated (aka a compactological space).
(*iii*) Show that every morphism $X \to Y$ to a compactological space Y factors uniquely via X^{qs}.