3.1.1. Exercise. Let $K > 0$ be a real number, and let $A \to B \to C$ be a complex of semi-normed abelian groups. Show that $A \to B \to C$ is K-normed exact if and only if the complex of completed semi-normed abelian groups $\hat{A} \to \hat{B} \to \hat{C}$ is K-normed exact. (We saw the forward direction in the lecture.)

3.1.2. Exercise. Let $f\colon V \to W$ be a surjective continuous linear map between two Banach spaces. Show that f is an epimorphism when considered as morphism of condensed sets. (Hint: denote the kernel of f by V_0, and show for profinite sets S that the sequence $C(S, V_0^\delta) \to C(S, V^\delta) \to C(S, W^\delta) \to 0$ of locally constant functions is K-normed exact for some K, by using the Banach open mapping theorem for f.)

3.1.3. Exercise. Let $f\colon V \to W$ be a surjective continuous linear map between two Banach spaces, and let K be a compact Hausdorff space. Show that the induced map $C(K, V) \to C(K, W)$ is surjective. (Hint: Use the preceding exercise and **??**.)

3.1.4. Exercise. Prove the Bartle–Graves theorem: every surjective continuous linear morphism of Banach spaces admits a continuous section. (This exercise had an incorrect hint. It is not clear how to use current ingredients to prove this theorem.)

3.1.5. Exercise. Compute $H^*(\mathbb{R}^n, M)$ for M a discrete abelian group. Hint: Use Theorem 1 from yesterday, and that $\mathbb{R}^n = \operatorname{colim}_{N\in\mathbb{N}}[-N, N]^n$. But beware: In general,

$$\operatorname{Ext}^i(\operatorname*{colim}_{N\in\mathbb{N}} A_N, B) \neq \lim_{N\in\mathbb{N}} \operatorname{Ext}^i(A_N, B).$$

It is true, however, if $\lim^1_{N\in\mathbb{N}} \operatorname{Ext}^i(A_N, B) = 0$ for every i. Read about $\lim^1$ if this is new to you.

3.1.6. Exercise. Use the preceding exercise and Breen–Deligne resolutions to prove Theorem 5: $\operatorname{Ext}^i(\mathbb{R}^n, M) = 0$ for all discrete abelian groups M and all i.

3.1.7. Exercise. We have a short exact sequence in Cond(Ab)

$$0 \to \mathbb{Z}^n \to \mathbb{R}^n \to (\mathbb{R}/\mathbb{Z})^n \to 0.$$

Use this along with Theorem 5 to compute

$$\operatorname{Ext}^i((\mathbb{R}/\mathbb{Z})^n, M) = \begin{cases} M^{\oplus n} & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

3.1.8. Exercise. Extend the preceding result to the case of infinite products

$$\operatorname{Ext}^i((\mathbb{R}/\mathbb{Z})^I, M) = \begin{cases} M^{\oplus I} & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

by using the Breen–Deligne resolution again. You will also need some spectral sequence comparison or derived category argument.

Now, we have assembled a good collection of Ext computations. We will use them to "compute" Ext groups between any two locally compact abelian groups.

3.1.9. Theorem. *Every locally compact abelian group A is of the form $A \cong \mathbb{R}^n \oplus A'$, where A' is an extension of a discrete group by a compact group.*

3.1.10. Theorem (Pontryagin duality)**.** *The functor* $A \mapsto \mathrm{Hom}(A, \mathbb{R}/\mathbb{Z})$ *is an antiequivalence of the category of locally compact abelian groups, which interchanges discrete and compact groups, and also interchanges closed embeddings and quotients by closed subgroups.*

You will prove that, for any locally compact abelian groups A and B,

$$(*) \qquad \mathrm{Ext}^i(A, B) = 0 \quad \text{for } i \geq 2$$

(just like in the category of ordinary abelian groups).

3.1.11. Exercise. Reduce to the case that each of A and B is either discrete, $\mathbb{R}^n$, or compact, using the structure theorem and long exact sequences of Ext groups.

3.1.12. Exercise. Prove that if A is a compact abelian group, then there is a short exact sequence of the form

$$0 \to A \to (\mathbb{R}/\mathbb{Z})^J \to (\mathbb{R}/\mathbb{Z})^I \to 0$$

in Cond(Ab). Hint: Use Pontryagin duality, and what you know about discrete abelian groups.

3.1.13. Exercise. Using the short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$, compute $\mathrm{Ext}^*(\mathbb{R}^n, \mathbb{R}/\mathbb{Z})$.

3.1.14. Exercise. Using all of the above results together with the theorems proved in today's lectures, show that $\mathrm{Ext}^*(A, B)$ is concentrated in degrees 0 and 1 in each of the nine cases: A is discrete, $\mathbb{R}^n$, or compact, and B is discrete, $\mathbb{R}^m$, or compact.