1.5.1. Exercise.

(i) (This exercise shows that metric spaces are sequential.) Let $f: X \to Y$ be a function between metric spaces, and let $x \in X$ be a point. Prove that f is continuous at x if and only if the following condition holds: for all sequences $\alpha: \mathbb{N} \to X$ that converge to x the image sequence $f \circ \alpha$ converges to $f(x)$.

(ii) Deduce that $Y \mapsto \underline{Y}$ is a fully faithful functor from metric spaces to condensed sets.

(iii) Assume that X has the Heine–Borel property: a subset of X is closed and bounded if and only if it is compact. Conclude that there is a natural compactology on X in which the small sets are precisely the bounded subsets of X.

1.5.2. Exercise. Let S be a set endowed with a bornology.

(i) Show that there exists a compactological space $\gamma(S)$ and a small function $S \to \gamma(S)$ with the following universal property: every small function $S \to X$ to a compactological space X extends uniquely to a morphism of compactological spaces $\gamma(S) \to X$.

(ii) If every subset of S is small, then $\gamma(S) \cong \beta(S)$.

1.5.3. Exercise. A morphism of compactological spaces is *strongly surjective* if every small subset of the codomain is the image of a small subset in the domain.

(i) Observe that a strongly surjective morphism is surjective.

(ii) Show that a morphism of compactological spaces is strongly surjective if and only if it is an surjection of condensed sets.

1.5.4. Exercise. Let X be a compact Hausdorff space, and view X as a compactological space by endowing it with its natural bornology (every set is small). Let $X^{(\omega)}$ denote the compactological space whose underlying topological space is X, but endowed with the bornology in which countable closed sets are small. Observe that

(i) the identity function $X^{(\omega)} \to X$ is a morphism;

(ii) if X is uncountable, this morphism is not strongly surjective;

(iii) if X is uncountable, then $X^{(\omega)}$ is not quasicompact (in the categorical sense; see below).

1.5.5. Exercise. Show that the essential image of the natural functor

$$\mathrm{Cond(Set)} \to [\mathrm{Extr}^{\mathrm{op}}, \mathrm{Set}]$$

consists of the full subcategory on those functors that send finite disjoint unions of extremally disconnected sets to finite products. In other words, the functors that satisfy the following two conditions:

(i) $X(\varnothing) = *$.

(ii) The natural map $X(S_1 \sqcup S_2) \to X(S_1) \times X(S_2)$ is an isomorphism for any two extremally disconnected sets S_1 and S_2.

1.5.6. Exercise.

(i) Show that an arbitrary coproduct of projective objects is projective.

(ii) Show that a retract of projective objects is projective.

(iii) Conclude that every projective condensed set is a retract of a coproduct of spaces of the form $\beta(S)$.

1.5.7. Exercise. Let S be a set that contains at least two elements. Show that the evaluation functor at $\beta(S)$

$$\begin{aligned} \mathrm{Cond}(\mathrm{Set}) &\longrightarrow \mathrm{Set} \\ X &\longmapsto X(\beta(S)) \end{aligned}$$

does not preserve binary coproducts.

1.5.8. Exercise. Here is the definition of quasiseparated (and quasicompact) from SGA4, specialized to condensed sets.

- A condensed set X is *quasicompact* if for any cover $\coprod Y_\alpha \twoheadrightarrow X$, some finite collection of the Y_α already cover X.
- A condensed set X is *quasiseparated* if for any diagram

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

with Y and Z quasicompact, the fiber product $Y \times_X Z$ is also quasicompact.

(i) Show that a condensed set X is quasicompact if and only if it can be covered by a single compact Hausdorff space.

(ii) Show that a condensed set X is quasicompact and quasiseparated if and only if it is a compact Hausdorff space.

(iii) Show that X is quasiseparated if and only if it is separated (the diagonal is closed). Hint for the proof: First prove that if K is a compact Hausdorff space and W is a quasicompact subobject of K, then W is also a compact Hausdorff space.

1.5.9. Exercise. Show that a condensed subobject of a compactological space is quasiseparated (aka a compactological space).

1.5.10. Exercise. This exercise constructs the "Hausdorffification" of a condensed set. In other words: we show that compactological spaces form a reflective subcategory of condensed sets. Let X be a condensed set.

(i) Show that there is a smallest closed equivalence relation $\bar{\Delta} \subseteq X \times X$. (You will need to use/prove that an arbitrary intersection of closed subobjects is closed.)

(ii) Show that the quotient X^{qs} of X by the equivalence relation $\bar{\Delta}$ is quasiseparated (aka a compactological space).

(iii) Show that every morphism $X \to Y$ to a compactological space Y factors uniquely via X^{qs}.