SEMINAR ON PILA–WILKIE POINT COUNTING AND APPLICATIONS

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Abstract. **Keywords:** o-minimality, diophantine geometry, detailed proofs

Tl;dr: In this seminar, we will refine our understanding of o-minimal theory by studying in detail the proof of the Pila–Wilkie point counting theorem. After that, we will look at applications of this result in diophantine geometry.

Main reference: We will follow a survey article by Scanlon [2].

Appetizer

Consider the following statement (a special case of Manin–Mumford):

0.1. **Theorem.** *Let $n > 0$ be a natural number, and let $\mathbb{G} = (\mathbb{C}^*)^n$ be the n-th power of the unit group of the complex numbers. Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a polynomial in n variables. Then the set*

$$X = \{(\zeta_1, \dots, \zeta_n) \in \mathbb{G} \mid \text{each } \zeta_i \text{ is a root of unity and } f(\zeta_1, \dots, \zeta_n) = 0\}$$

is a finite union of cosets of subgroups of $\mathbb{G}$.

Originally, this statement was proven by Mann, but we will be interested in the Pila–Zannier argument. It goes as follows. Let $\exp\colon \mathbb{C} \to \mathbb{C}^*$ denote the function $z \mapsto e^{2\pi i z}$. Observe that there is an analytic covering $\exp\colon \mathbb{C}^n \to \mathbb{G}$. A tuple $\zeta = (\zeta_1, \dots, \zeta_n)$ consists of roots of unity if and only if there is some rational $a \in \mathbb{Q}^n$ such that $\exp(a) = \zeta$. This means that we can translate our problem into a question about rational solutions to the transcendental equation $f(\exp(z)) = 0$.

It may seem as if we have made the problem a lot more difficult. However, by restricting to a fundamental domain

$$D = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid 0 \le \mathrm{Re}(z_i) < 1 \text{ for each } i\}$$

we end up in a *tame* situation.

- (i) It is again the case that $\zeta \in \mathbb{G}$ is a tuple of roots of unity if and only if there exists a *rational* $a \in D \cap \mathbb{Q}^n$ such that $\exp(a) = \zeta$.
- (ii) The restriction of exp to D is a definable function in the structure $\mathbb{R}_{\mathrm{an,exp}}$, which is an *o-minimal* structure. From the point of view of mathematical logic, this means it is exceedingly well-behaved.

We may now consider the set

$$\tilde{X} = \{z \in D \mid f(\exp(z)) = 0\}$$

which is an example of a *definable set* in $\mathbb{R}_{\mathrm{an,exp}}$. This is where the Pila–Wilkie point counting theorem comes in.

Understanding the rational solutions to algebraic equations is a notoriously hard problem. But it turns out that one can get a good grip on rational solutions to *transcendental* equations.

Let $Y \subseteq \mathbb{R}^m$ be any set, definable in some o-minimal expansion of the reals. We define the *algebraic part* $Y^{\mathrm{alg}} \subseteq Y$ to be the union of all connected, positive dimensional semialgebraic subsets of Y. (Recall that a set is semialgebraic if it is definable using Boolean combinations of polynomial inequalities.) Next, we define the *transcendental part* of Y to be $Y - Y^{\mathrm{alg}}$.

The Pila–Wilkie point counting theorem asserts that there are sub-exponentially many rational points in Y^{tra}. To make this precise, we introduce the following function, which counts rational points of *bounded height*:

$$N(Y,t) = \#\{(\tfrac{a_1}{b_1}, \dots, \tfrac{a_n}{b_n}) \in Y^{\mathrm{tra}} \mid \text{for each } i \text{ we have } |a_i| \leq t, |b_i| \leq t, a_i, b_i \in \mathbb{Z}\}$$

0.2. **Theorem** (Pila–Wilkie)**.** *For each $\epsilon > 0$ there is a constant $C = C_\epsilon$ so that $N(Y,t) \leq Ct^\epsilon$ for all $t \geq 1$.*

Now we return to our definable set of interest: $\tilde{X}$. Using a result by Ax (a function field version of the Schanuel conjecture) we can show that $\tilde{X}^{\mathrm{alg}}$ is indeed a finite union of cosets of subgroups (intersected with D).

Hence we are done if we show that $\tilde{X}^{\mathrm{tra}}$ only has finitely many rational points. This is done by contradiction. If there are infinitely many points, then one can use Galois theory to show that $N(Y,t)$ must exhibit exponential growth. This contradicts Pila–Wilkie, so we win.

1. Introduction

Give an overview of the seminar. A long form of the appetizer above, with a bit more details on what is meant with an o-minimal structure.

Besides Manin–Mumford for tori, mention applications to Manin–Mumford for abelian varieties as well as the André–Oort conjecture for Shimura varieties.

Reference. §1 and §2 (aka p1–5) of [2]

2. O-minimality I: crash course on logic

Recap: language, structure, definable set, formula, sentence, theory, model.

In particular, recap the notation $\mathfrak{M} \models \varphi$.

State compactness! Explain what it means!

Reference. §3.1 (aka p6–10) of [2]

3. O-minimality II: examples

State theorems of Tarski and Wilkie. Discuss proof of Tarski's theorem in some detail?

Explain what it means that $\mathbb{R}_{\mathrm{exp}}$ is model complete, but don't discuss the proof of Wilkie's theorem.

State theorem by Van den Dries, that $\mathbb{R}_{\mathrm{an}}$ is o-minimal.

Finally, discuss $\mathbb{R}_{\mathrm{an,exp}}$.

Reference. first half of §3.2 (aka p10–14) of [2]

4. O-minimality III: cell decomposition

Prove the existence of Skolem functions.

Define cells, state cell decomposition. Discuss the proof. (Note: a detailed proof is long and complicated.)

Derive Lemma 3.35 of [2] on non-constant definable analytic curves in definable sets.

Reference. second half of §3.2 (aka p14–18) of [2]

5. Pila–Wilkie I: overview and structure of proof

Define the multiplicative height of rational numbers and the point counting function.

Define the algebraic part and the transcendental part of a definable set.

State the Pila–Wilkie theorem.

State the two main ingredients in the proof: Theorem 4.8 and Theorem 4.31 (from §4.3)

State and prove Proposition 4.11 (uses compactness, take your time) and Proposition 4.15.

Reference. §4.1 of [2].

6. Pila–Wilkie II: parametrization theorem

The goal is to prove Theorem 4.8.

State Theorem 4.19. Carefully outline the induction strategy.

Carry out the proof.

Reference. Last bit of §4.1 and all of §4.2 of [2].

7. Pila–Wilkie III: diophantine approximation

The goal is to prove Theorem 4.31.

Introduce notations, and state and prove Propositions 4.28 and 4.30.

This section includes a lot of complicated formulas. Think carefully about the best way to present this material.

Reference. §4.3 of [2].

8. Pila–Wilkie IV: end of proof, refinements

Recap the proof structure: recall statement of Pila–Wilkie, Proposition 4.11, and Theorem 4.31.

Tie everything together: State Proposition 4.33

State Theorem 4.32 (generalization of Pila–Wilkie) and prove it.

Talk about refinements.

Reference. §4.4 of [2].

9. Applications I: Manin–Mumford for tori

Make the appetizer precise.

Reference. See §5 of [1] for detailed calculations.

10. Applications II: Manin–Mumford for abelian varieties

The goal is to state and prove Theorem 5.1.

Along the way state and assume Theorems 5.5 and 5.7.

Reference. §5.1 of [2].

References

[1] Thomas Scanlon. "Counting special points: logic, Diophantine geometry, and transcendence theory". In: *Bull. Amer. Math. Soc. (N.S.)* 49.1 (2012), pp. 51–71.

[2] Thomas Scanlon. "O-minimality as an approach to the André-Oort conjecture". In: *Around the Zilber-Pink conjecture/Autour de la conjecture de Zilber-Pink*. Vol. 52. Panor. Synthèses. Soc. Math. France, Paris, 2017, pp. 111–165.