Liquid Tensor Experiment

Credit: https://twitter.com/Jcrudess/status/1338922029278441483/photo/1

1998 Liquid Tension Experiment

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1999 Liquid Tension Experiment 2

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2020 Dec 05: “Liquid Tensor Experiment”, Peter Scholze

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2020 Dec 14: announcement LTE 3

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bonus track: *Solid Resolution Theory*

Liquid analytic ring structure on $\mathbb{R}$

Analytic geometry

Schemes, adic spaces, manifolds/$\mathbb{C}$

Definition

An *analytic ring* $\mathcal{A}$ is a condensed ring $\underline{\mathcal{A}}$

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together with a functor

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giving a "free complete $\underline{\mathcal{A}}$-module" on S.

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Let $\mathcal{M}_{p'}(S)$ be the space of p'-measures on S.

Then

$$\mathrm{Ext}^i_{\mathrm{Cond(Ab)}}(\mathcal{M}_{p'}(S), V) = 0$$

for $i \geqslant 1$.

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- "I think this may be my most important theorem to date"

First target (Thm 9.4 of Analytic.pdf)

Fix $0 < r < r' < 1$.

For any m, there exists a k and c_0
such that for all profinite sets S
and r-normed $\mathbb{Z}[T^{\pm 1}]$-modules V
the system of complexes

$$C_c^\bullet \colon \hat{V}(\mathcal{M}_{r'}(S)_{\leqslant c})^{T^{-1}} \to \hat{V}(\mathcal{M}_{r'}(S)^2_{\leqslant \kappa_1 c})^{T^{-1}} \to \dots$$

is $\leqslant k$-exact in degrees $\leqslant m$ for $c \geqslant c_0$.

(Question 9.9 of Analytic.pdf)

How does k grow, in terms of r, r', m, etc?

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- Alternative to Breen–Deligne resolutions

Lean theorem prover

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mathlib, main mathematical library

Lean demo

Theorem (Breen–Deligne)

There exists a functorial resolution of an abelian group A of the form

$$\cdots \to \bigoplus_{j=1}^{n_j} \mathbb{Z}[A^{r_{i,j}}] \cdots \to \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0$$

where all n_j and $r_{i,j}$ are natural numbers.

Lemma

There exists a completely explicit functorial *complex* of an abelian group A of the form

$$C(A): \quad \cdots \to \mathbb{Z}[A^{2^i}] \cdots \to \mathbb{Z}[A^4] \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0$$

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then $\mathrm{Ext}^i(A, B) = 0$ for all $i \geqslant 0$.

Joint work with

- Peter Scholze

The Lean community

- Adam Topaz
- Riccardo Brasca
- Patrick Massot
- Scott Morrison
- Kevin Buzzard
- Bhavik Mehta
- Damiano Testa
- Filippo A.E. Nuccio
- Heather Macbeth
- Mario Carneiro

Talk 2: Lean

Talk 3: math

Pointers

Community website

`leanprover-community.github.io`

Chatroom

`leanprover.zulipchat.com`

Natural number game

`wwwf.imperial.ac.uk/~buzzard/xena/natural_number_game/`

THE END