LIQUID STRUCTURE ON THE REALS — PROOF SKETCH

JOHAN COMMELIN

Contents

1. Prolegomena

1.1. This text presents an outline of the proof by Clausen and Scholze, that there is an analytic ring structure on the real numbers. The main sources are the lecture notes [1, 2].

The aim is not to cover all the details of the proof, but to give an overview of the structure, and to motivate some of the surprising steps. I hope that this text can serve as a roadmap through for the actual proof, as written up in [2].

I thank Peter Scholze for his patient explanations and corrections.

1.2. This text will not recall the basics of the condensed formalism; see [1] for details.

1.3. In this text we write $\lim D$ (resp. $\operatorname{colim} D$) for the categorical *limit* (resp. *colimit*) of a diagram D. Reminder: $\lim D = \varprojlim D$ and $\operatorname{colim} D = \varinjlim D$.

2. Analytic rings

2.1. This section recalls the definition and basic properties of analytic rings. The main sources are [1, §VII] and [2, App. to §VI].

Proposition 2.5 is a crucial lemma that we will need later on.

2.2. **Definition** ([1, Def. 7.1, 7.4]). (*i*) A *pre-analytic ring* is a pair $(\mathcal{A}, \mathcal{M})$ consisting of a condensed ring $\mathcal{A}$ together with a functor

$$\mathcal{M}\colon \{\text{extr. disc. sets}\} \longrightarrow \mathrm{Mod}_{\mathcal{A}}^{\mathrm{cond}}$$
$$S \longmapsto \mathcal{M}[S]$$

to the category of $\mathcal{A}$-modules in condensed abelian groups, taking finite disjoint unions to finite products, and a natural transformation $S \to \mathcal{M}[S]$.

(*ii*) An *analytic ring* $(\mathcal{A}, \mathcal{M})$ is a pre-analytic ring with the following property: for every complex

$$C\colon \quad \cdots \to C_i \to \cdots \to C_2 \to C_1 \to C_0$$

of $\mathcal{A}$-modules in condensed abelian groups for which all C_i are direct sums of modules of the form $\mathcal{M}[T]$ for varying extremally disconnected sets T, the map

$$R\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{M}[S], C) \longrightarrow R\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}[S], C)$$

is an isomorphism of condensed abelian groups for all extremally disconnected sets S.

2.3. (*i*) The module $\mathcal{M}[S]$ should be thought of as the "free complete" module generated by S. But it should be stressed that $\mathcal{M}$ is additional data, and different choices can be made for a given condensed ring $\mathcal{A}$.

(*ii*) For the time being, we will not be concerned with applications of this definition. Suffice it to say that there is a natural way to define the category of modules over an analytic ring, and this category has very good properties. For details, see [1, §VII].

For other discussions of applications, see the video recordings of the masterclass on condensed mathematics, held in Copenhagen, 2020.

(*iii*) Our goal is to construct a particular example of an analytic ring: namely the liquid structure on the real numbers.

2.4. **Lemma.** *A pre-analytic ring $(\mathcal{A}, \mathcal{M})$ is an analytic ring if the following condition is satisfied: for all index sets I and J, and extremally disconnected sets S_i, S'_j and all maps*

$$f\colon \bigoplus_i \mathcal{M}[S_i] \longrightarrow \bigoplus_j \mathcal{M}[S'_j]$$

of condensed $\mathcal{A}$-modules with kernel K, the map

$$R\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{M}[S], K) \longrightarrow R\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}[S], K)$$

is an isomorphism in $D(\mathrm{Cond}(\mathrm{Ab}))$, for all extremally disconnected sets S.

Proof. See [2, Rem. 6.13]: Write the complex C occuring in the definition of analytic ring as a limit of its Postnikov truncations. □

2.5. **Proposition** ([2, Prop. 6.15])**.** *Let $(\mathcal{A}, \mathcal{M})$ be an analytic ring. The module $\mathcal{A}' \stackrel{\text{def}}{=} \mathcal{M}[*]$ is a condensed ring, and for every S, $\mathcal{M}[S]$ is naturally a module over $\mathcal{A}'$. This yields a natural morphism of pre-analytic rings $(\mathcal{A}, \mathcal{M}) \to (\mathcal{A}', \mathcal{M})$.*

The pre-analytic ring $(\mathcal{A}', \mathcal{M})$ is analytic, and the forgetful functor $D(\mathrm{Mod}^{\mathrm{cond}}_{(\mathcal{A}',\mathcal{M})}) \to D(\mathrm{Mod}^{\mathrm{cond}}_{(\mathcal{A},\mathcal{M})})$ is an equivalence.

3. Spaces of measures I

3.1. In this section we define certain spaces of measures $\mathcal{M}_{<p}(S)$, depending on a real number $0 < p \leq 1$. The functor $S \mapsto \mathcal{M}_{<p}(S)$ will define the liquid analytic ring structure on $\mathbb{R}$.

Let $0 < p \leq 1$ be a real number.

3.2. **Definition.** Let S be a finite set. The condensed $\mathbb{R}$-module $\mathbb{R}[S]$ is naturally filtered:

$$\mathbb{R}[S]_{\ell^p \leq c} = \left\{ (a_s)_s \in \mathbb{R}[S] \;\middle|\; \sum_s |a_s|^p \leq c \right\}$$

This construction is functorial in S.

3.3. **Definition.** For any profinite set S, written as limit of finite sets $S = \lim_i S_i$, we define
$$\mathcal{M}_p(S)_{\le c} = \lim_i \mathbb{R}[S_i]_{\ell^p \le c}.$$
This construction is functorial in S.

The colimit
$$\mathcal{M}_p(S) = \bigcup_{c \ge 0} \mathcal{M}_p(S)_{\le c}$$
is naturally a condensed $\mathbb{R}$-module.

3.4. **Definition.** For $p' < p$, there are natural maps
$$\mathcal{M}_{p'}(S) \to \mathcal{M}_p(S)$$
and we define
$$\mathcal{M}_{<p}(S) = \operatorname*{colim}_{p'<p} \mathcal{M}_{p'}(S).$$
The space $\mathcal{M}_{<p}(S)$ is a condensed $\mathbb{R}$-module, functorial in S.

4. Synopsis: main statement

4.1. **Theorem.** *Let $0 < p \le 1$ be a real number. Then*
$$(\mathbb{R}, S \mapsto \mathcal{M}_{<p}(S))$$
is an analytic ring.

4.2. The first step towards proving this theorem is to construct a pre-analytic ring of overconvergent Laurent series, which admits $(\mathbb{R}, S \mapsto \mathcal{M}_{<p}(S))$ as quotient.

By showing that the ring of overconvergent Laurent series is an analytic ring, we may then conclude that the real numbers are also an analytic ring. This is carried out in Section 7.

One may argue that the liquid structure on the ring of overconvergent Laurent series is actually the main result. I expect that this analytic ring will play an important role in liquid mathematics in the future.

5. Convergent Laurent series

5.1. Several subrings of $\mathbb{Z}((T))$ will play an important role in the proof. We will introduce these subrings in this section. See also [2, §VII].

5.2. **Definition.** Let $r > 0$ be a real number. Denote by $\mathbb{Z}((T))_r$ the following subring of $\mathbb{Z}((T))$:
$$\Big\{ \sum_{n \gg -\infty} a_n T^n \,\Big|\, \sum |a_n| r^n < \infty \Big\}$$
which is naturally filtered by
$$\mathbb{Z}((T))_{r,\le c} = \Big\{ \sum_{n \gg -\infty} a_n T^n \,\Big|\, \sum |a_n| r^n \le c \Big\}.$$

5.3. (*i*) It is crucial for later steps in the proof that $\mathbb{Z}((T))_{r,\le c}$ is naturally a profinite set. We leave this verification as exercise to the reader (or see [2, Prop. 6.8]).

(*ii*) Note that for $r \ge 1$, the subring $\mathbb{Z}((T))_r$ consists exactly of the Laurent polynomials. In general, $\mathbb{Z}((T))_{r'} \subseteq \mathbb{Z}((T))_r$ for $r' \ge r$.

5.4. **Definition.** The ring of overconvergent Laurent series $\mathbb{Z}((T))_{>r}$ is the union $\bigcup_{r'>r} \mathbb{Z}((T))_{r'}$. Equivalently, it is

$$\Big\{ \sum_{n \gg -\infty} a_n T^n \,\Big|\, \exists r' > r, |a_n|(r')^n \to 0 \Big\}.$$

5.5. The ring $\mathbb{Z}((T))_{>r}$ is quite spectacular. It is a principal ideal domain, whose quotients consist of almost all the local fields that show up in number theory:

(*i*) $\mathbb{R}$ and $\mathbb{C}$;
(*ii*) finite extensions of $\mathbb{Q}_p$, for all p;
(*iii*) $\mathbb{F}_p((T))$, for all p.

The reader is encouraged to read more about the details in [2, §VII].

Pick some $0 < x \le r$. We will focus on the quotient

$$\begin{aligned} \theta_x \colon \mathbb{Z}((T))_r &\longrightarrow \mathbb{R} \\ T &\longmapsto x \end{aligned}$$

For our applications, we can always arrange $x = \frac{1}{2}$, in which case it is clear that $\ker(\theta_x)$ is a principal ideal, namely $(2T-1) = (T^{-1}-2)$.

We get a similar quotient map $\mathbb{Z}((T))_{>r} \to \mathbb{R}$. What is spectacular, is that $\mathbb{R}$ is not just a quotient of $\mathbb{Z}((T))_{>r}$ as ring, but also as pre-analytic ring, see Proposition 6.5 below.

6. Spaces of measures II

6.1. In this section we define certain spaces of measures with coefficients in $\mathbb{Z}((T))_{>r}$, for $0 < r < 1$. There are many similarities with Section 3, but I have not found a precise common generalisation.

The main result is Proposition 6.5, which asserts that the pre-analytic ring $(\mathbb{R}, S \mapsto \mathcal{M}_{<p}(S))$ is a quotient of a pre-analytic ring structure on $\mathbb{Z}((T))_{>r}$.

Fix some $0 < r < 1$.

6.2. **Definition.** Let S be a finite set. The condensed $\mathbb{Z}((T))_r$-module $\mathbb{Z}((T))_r[S]$ is naturally the union of:

$$\mathbb{Z}((T))_r[S]_{\le c} = \Big\{ \sum_{n \in \mathbb{Z}, s \in S} a_{n,s} T^n[s] \in \mathbb{Z}((T))_r[S] \,\Big|\, a_{n,s} \in \mathbb{Z}, \sum |a_{n,s}| r^n \le c \Big\}.$$

This construction is functorial in S.

6.3. **Definition.** For any profinite set S, written as limit of finite sets $S = \lim_i S_i$, we define

$$\mathcal{M}(S, \mathbb{Z}((T))_r)_{\le c} = \lim_i \mathbb{Z}((T))_r[S_i]_{\le c}.$$

This construction is functorial in S.

The colimit

$$\mathcal{M}(S, \mathbb{Z}((T))_r) = \bigcup_{c \ge 0} \mathcal{M}(S, \mathbb{Z}((T))_r)_{\le c}$$

is naturally a condensed $\mathbb{Z}((T))_r$-module.

6.4. **Definition.** For $r' > r$, there are natural maps

$$\mathcal{M}(S, \mathbb{Z}((T))_{r'}) \to \mathcal{M}(S, \mathbb{Z}((T))_r)$$

and we define

$$\mathcal{M}(S, \mathbb{Z}((T))_{>r}) = \operatorname*{colim}_{r' > r} \mathcal{M}(S, \mathbb{Z}((T))_{r'}).$$

The space $\mathcal{M}(S, \mathbb{Z}((T))_{>r})$ is a condensed $\mathbb{Z}((T))_{>r}$-module, functorial in S.

6.5. **Proposition.** *Pick $0 < x \leq r < 1$, and let $0 < p \leq 1$ be such that $x^p = r$. Recall the map $\theta_x \colon \mathbb{Z}((T))_r \to \mathbb{R}$ from 5.5. Then there are natural isomorphisms*

$$\mathcal{M}(S, \mathbb{Z}((T))_r)/\ker(\theta_x) \cong \mathcal{M}_p(S)$$
$$\mathcal{M}(S, \mathbb{Z}((T))_{>r})/\ker(\theta_x) \cong \mathcal{M}_{<p}(S)$$

Proof. The ingredients for a proof are in [2, Prop. 7.2]. □

7. Synopsis: reduction to an arithmetic setting

7.1. In order to prove Theorem 4.1, it suffices to prove the following theorems.

7.2. **Theorem.** *Let $0 < r < 1$ be a real number. Then*

$$(\mathbb{Z}[T^{-1}], S \mapsto \mathcal{M}(S, \mathbb{Z}((T))_{>r}))$$

is an analytic ring.

7.3. **Theorem.** *Let $0 < r < 1$ be a real number. Then*

$$(\mathbb{Z}((T))_{>r}, S \mapsto \mathcal{M}(S, \mathbb{Z}((T))_{>r}))$$

is an analytic ring.

Proof. Apply Proposition 2.5 to Theorem 7.2. □

Proof of Theorem 4.1. Put $r = (\frac{1}{2})^p$. As argued in Proposition 6.5, the pre-analytic ring $(\mathbb{R}, S \mapsto \mathcal{M}_{<p}(S))$ is a quotient of $(\mathbb{Z}((T))_{>r}, S \mapsto \mathcal{M}(S, \mathbb{Z}((T))_{>r}))$. Since the latter is an analytic ring by Theorem 7.3, we conclude that the pre-analytic ring structure on the reals is also an analytic ring (see also [2, Prop. 12.8]). □

7.4. We are now left with the formidable task of proving Theorem 7.2. It will turn out to be crucial that we use the base ring $\mathbb{Z}[T^{-1}]$, which is in some sense just small enough to make the argument work.

Specifically, to move forward we will prove the following more explicit statement, which is sufficient by Lemma 2.4. However, the first thing we do after stating it, is to carry out a sequence of further reduction steps.

7.5. **Proposition.** *Let K be a condensed $\mathbb{Z}[T^{-1}]$-module that is the kernel of a map*

$$f \colon \bigoplus_{i \in I} \mathcal{M}(S_i, \mathbb{Z}((T))_{>r}) \longrightarrow \bigoplus_{j \in J} \mathcal{M}(S'_j, \mathbb{Z}((T))_{>r})$$

where all S_i and S'_j are extremally disconnected. Then for all $1 > r' > r$ and all profinite S, the map

$$R\underline{\mathrm{Hom}}_{\mathbb{Z}[T^{-1}]}(\mathcal{M}(S, \mathbb{Z}((T))_{r'}), K) \to R\underline{\mathrm{Hom}}_{\mathbb{Z}[T^{-1}]}(\mathbb{Z}[T^{-1}][S], K)$$

in $D(\mathrm{Cond}(\mathrm{Ab}))$ is an isomorphism.

8. Simplifying the target of $R\underline{\mathrm{Hom}}$

8.1. We present here a summary of the reduction steps made in [2, §VIII]. Beware: the details take up more than 7 pages. We are sweeping a lot under the rug.

To approach Proposition 7.5, the first step is to drastically simplify the condensed $\mathbb{Z}[T^{-1}]$-modules K for which we need to verify the claim of that Proposition.

8.2. The starting point of the simplifications is [2, Prop. 8.3], which shows that $\mathcal{M}(S, \mathbb{Z}((T))_r$ is a pseudocoherent condensed $\mathbb{Z}[T^{-1}]$-module.

The upshot is that $R\underline{\mathrm{Hom}}_{\mathbb{Z}[T^{-1}]}(\mathcal{M}(S, \mathbb{Z}((T))_r, K)$ commutes with filtered colimits in K. In a short sequence of steps, this allows one to get rid of the direct sums in Proposition 7.5.

8.3. The next steps make a very careful analysis of the objects K for which one still has to verify the claim of Proposition 7.5. These are carried out in Proposition 8.5 through Theorem 8.12 of [2].

These allow us to identify some sort of "Banach spaces" V inside the remaining K, and it suffices to check the claims for these spaces V. The precise characterisation of these V is given by the following definition.

8.4. **Definition** ([2, Def. 8.13])**.** An r-normed $\mathbb{Z}[T^{\pm 1}]$-module is a $\mathbb{Z}[T^{\pm 1}]$-module V that is semi-normed (as abelian group) satisfying $\|Tv\| = r\|v\|$ for all $v \in V$.

8.5. We will see later on, that r-normed modules are amenable to resolutions by profinite sets. This is one useful ingredient in the final proof.

To such an r-normed $\mathbb{Z}[T^{\pm 1}]$-module V, we attach a condensed abelian group $\hat{V}$:

$$\begin{aligned} \hat{V} \colon \mathrm{Profinite} &\longrightarrow \mathrm{Ab} \\ S &\longmapsto \text{completion of } \{f \colon S \to V \mid f \text{ is loc. constant}\} \end{aligned}$$

Note that $\hat{V}$ is naturally a condensed $\mathbb{Z}[T^{\pm 1}]$-module, and $\hat{V}(S)$ is a complete semi-normed group endowed with the sup-norm, for every profinite set S.

All in all, we are now left with the following target. (In the process, the claim about internal homs has also been simplified to a claim about external homs.)

8.6. **Proposition** ([2, Thm 8.14])**.** *Fix radii $1 > r' > r > 0$. Then for all r-normed $\mathbb{Z}[T^{\pm 1}]$-modules V and all profinite sets S, the map*

$$R\mathrm{Hom}_{\mathbb{Z}[T^{-1}]}(\mathcal{M}(S, \mathbb{Z}((T))_{r'}), \hat{V}) \longrightarrow R\mathrm{Hom}_{\mathbb{Z}[T^{-1}]}(\mathbb{Z}[T^{-1}][S], \hat{V})$$

is a quasi-isomorphism.

9. Simplifying the source of $R\mathrm{Hom}$

9.1. We will now explain the reduction steps applied to the source of the $R\mathrm{Hom}$'s. One of the elements in the final proof is the so-called Breen–Deligne resolution. This tool is great for showing that some derived functor vanishes. So we will now move to such a setting.

9.2. **Definition.** Let $\overline{\mathcal{M}}_{r'}(S)$ denote the quotient

$$\mathcal{M}(S, \mathbb{Z}((T))_{r'})/\mathbb{Z}[T^{-1}][S].$$

9.3. This is a place where it is crucial that we work with the base ring $\mathbb{Z}[T^{-1}]$, because it allows us to give the following explicit description of the space $\overline{\mathcal{M}}_{r'}(S)$. We proceed in the same steps as the definition of $\mathcal{M}(S, \mathbb{Z}((T))_{r'})$.

$$\begin{aligned} \overline{\mathcal{M}}_{r'}(S)_{\le c} &= \Big\{ \sum_{n \ge 1, s \in S} a_{n,s} T^n [s] \,\Big|\, \sum_{n \ge 1, s \in S} |a_{n,s}| (r')^n \le c \Big\} && S \text{ finite} \\ \overline{\mathcal{M}}_{r'}(S)_{\le c} &= \lim_i \overline{\mathcal{M}}_{r'}(S_i)_{\le c} && S = \lim_i S_i \text{ profinite} \\ \overline{\mathcal{M}}_{r'}(S) &= \bigcup_{c \ge 0} \overline{\mathcal{M}}_{r'}(S)_{\le c} \end{aligned}$$

Crucially, the spaces $\overline{\mathcal{M}}_{r'}(S)_{\le c}$ are profinite.

By using the short exact sequence

$$0 \to \mathbb{Z}[T^{-1}][S] \to \mathcal{M}(S, \mathbb{Z}((T))_{r'}) \to \overline{\mathcal{M}}_{r'}(S) \to 0$$

we can therefore reduce our task to checking that

$$R\mathrm{Hom}_{\mathbb{Z}[T^{-1}]}(\overline{\mathcal{M}}_{r'}(S), \hat{V}) = 0$$

for all profinite sets S and r-normed $\mathbb{Z}[T^{\pm 1}]$-modules V.

9.4. **Theorem.** *Fix $0 < r < r' < 1$. For all profinite sets S, and all r-normed $\mathbb{Z}[T^{\pm 1}]$-modules V, we have*

$$R\mathrm{Hom}_{\mathbb{Z}[T^{-1}]}(\overline{\mathcal{M}}_{r'}(S), \hat{V}) = 0.$$

9.5. It is finally time to start computing the RHom. We will proceed by resolving $\overline{\mathcal{M}}_{r'}(S)$, using a functorial resolution known as the Breen–Deligne resolution, with the useful property that it is constructed completely from objects of the form $\mathbb{Z}[M]$. More on these resolutions below, see Section 11.

In our setting, the M's occuring in the objects $\mathbb{Z}[M]$ will be powers of $\overline{\mathcal{M}}_{r'}(S)$. We are interested in evaluating $\mathrm{Hom}_{\mathbb{Z}[T^{-1}]}(\mathbb{Z}[M], \hat{V})$. To proceed, we make two more steps.

First of all, we should keep in mind that T^{-1} acts on M, and therefore there is a natural map $\mathbb{Z}[T^{-1}][M] \to \mathbb{Z}[M]$. Now there are two different actions of T^{-1} on $\mathbb{Z}[T^{-1}][M]$: the first acts on the 'inside', via M, the second acts on the 'outside', via the coefficients $\mathbb{Z}[T^{-1}]$. We get a resolution

$$0 \to \mathbb{Z}[T^{-1}][M] \xrightarrow{T^{-1}-[T^{-1}]} \mathbb{Z}[T^{-1}][M] \to \mathbb{Z}[M] \to 0$$

We should therefore get a grip on

$$\mathrm{Hom}_{\mathbb{Z}[T^{-1}]}(\mathbb{Z}[T^{-1}][M], \hat{V}) = \hat{V}(M),$$

the map

$$\hat{V}(M) \xrightarrow{T^{-1}-[T^{-1}]^*} \hat{V}(M),$$

and its kernel. Denote this kernel by $\hat{V}(M)^{T^{-1}}$. Naively speaking, we would want to show that

$$\hat{V}(\overline{\mathcal{M}}_{r'}(S))^{T^{-1}} \to \hat{V}(\overline{\mathcal{M}}_{r'}(S)^2)^{T^{-1}} \to \dots$$

is exact.

However, M is not a profinite set. Therefore $\hat{V}(M)$ does not admit a direct description as completion of the group of locally constant functions $M \to V$. This is where the final step comes in. Recall that $\overline{\mathcal{M}}_{r'}(S)$ is a colimit of the profinite sets $\overline{\mathcal{M}}_{r'}(S)_{\le c}$. We will thus add another dimension to the (already 2-dimensional) resolution of $\overline{\mathcal{M}}_{r'}(S)$, by replacing $\hat{V}(M)$ with the diagram $(\hat{V}(M_{\le c}))_{c \in \mathbb{R}_{\ge 0}}$.

Summing up, We need to show that

$$\hat{V}(\overline{\mathcal{M}}_{r'}(S)_{\le \bullet})^{T^{-1}} \to \hat{V}(\overline{\mathcal{M}}_{r'}(S)^2_{\le \bullet})^{T^{-1}} \to \dots$$

is exact, for a suitable notion of exactness. We will now make this notion precise.

9.6. **Definition.** A *system of complexes* is a collection of complexes of semi-normed abelian groups $C_c^\bullet$, for $c \in \mathbb{R}_{\ge 0}$, with restriction maps $C_{c_1} \to C_{c_2}$, for $c_1 \ge c_2$, such that the obvious diagrams commute.

9.7. **Definition.** Fix a natural number m, and reals $k \geq 1$, $K \geq 0$, and $c_0 \geq 0$. A system of complexes $C^\bullet_\bullet$ is *weakly $\leq k$-exact in degrees $\leq m$ for $c \geq c_0$ with bound K* if the following condition is satisfied.

For all $c \geq c_0$, all $x \in C^i_{kc}$ with $i \leq m$ and any $\varepsilon > 0$, there is some $y \in C^{i-1}_c$ such that

$$\|x_{|c} - dy\| \leq K\|dx\| + \varepsilon.$$

9.8. The following result show the relation of this new concept with the ordinary notion of exactness.

9.9. **Lemma.** *Let $k \geq 1$, $c_0 \geq 0$ be real numbers, and $m \in \mathbb{N}$. Let $C^\bullet_\bullet$ be a system of complexes, and for each $c \geq 0$ let D_c be a cochain complex of semi-normed groups. Let $f_c \colon C^\bullet_{kc} \to D^\bullet_c$ and $g_c \colon D^\bullet_c \to C^\bullet_c$ be norm-nonincreasing morphisms of cochain complexes of semi-normed groups such that $g_c \circ f_c$ is the restriction map $C^\bullet_{kc} \to C^\bullet_c$. Assume that for all $c \geq c_0$ the cochain complex D_c is normed exact. Then $C^\bullet_\bullet$ is weakly $\leq k$-exact in degrees $\leq m$ and for $c \geq c_0$ with bound 1.*

10. Synopsis: a technical home straight

10.1. After a shear endless amount of simplification steps, we are now ready to state the technical theorem that marks the final episode of the grand proof.

The rest of this text covers material from [2, §IX].

In this section, we will often write M for $\overline{\mathcal{M}}_{r'}(S)$ or similar objects.

10.2. **Theorem.** *Fix $0 < r < r' < 1$. For each natural number m, there exist constants $k \geq 1$, $K \geq 0$, and $c_0 \geq 0$, such that for all profinite sets S, and r-normed $\mathbb{Z}[T^{\pm 1}]$-modules V, the system of complexes*

$$\hat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leq \kappa_0 c})^{T^{-1}} \to \hat{V}(\overline{\mathcal{M}}_{r'}(S)^2_{\leq \kappa_1 c})^{T^{-1}} \to \dots$$

is weakly $\leq k$-exact in degrees $\leq m$ and for $c \geq c_0$ with bound K.

10.3. The numbers $\kappa_0, \kappa_1, \dots$ appearing in the statement, are a sequence of constants in $\mathbb{R}_{\geq 0}$ depending on the chosen Breen–Deligne resolution. The reason they pop up is the following: As we will see, the differentials $\hat{V}(M^m)^{T^{-1}} \to \hat{V}(M^n)^{T^{-1}}$ showing up in the complex do not map $\hat{V}(M^m_{\leq c})^{T^{-1}}$ into $\hat{V}(M^n_{\leq c})^{T^{-1}}$, for all $c \in \mathbb{R}_{\geq 0}$. Instead, there exists a κ, such that $\hat{V}(M^m_{\leq c})^{T^{-1}}$ maps to $\hat{V}(M^n_{\leq \kappa c})^{T^{-1}}$. Inductively, this leads to the sequence of constants $(\kappa_i)_i$ appearing in the statement above.

10.4. The proof of Theorem 10.2 relies on several ideas, and many technical estimates. We summarize three of the main ideas, omitting many important details.

10.5. The first step is to construct a *system of double complexes*, by resolving $M_{\leq c}$ with the Čech cover of the summation map $\sum \colon M^N_{\leq c/N} \to M_{\leq c}$, for some suitable $N \in \mathbb{N}$, applying the $\hat{V}(\_)^{T^{-1}}$ construction, and taking alternating sums of the resulting maps coming from the Čech cover.

$$\begin{array}{ccccc}
\hat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leq \kappa_0 c})^{T^{-1}} & \longrightarrow & \hat{V}(\overline{\mathcal{M}}_{r'}(S)^2_{\leq \kappa_1 c})^{T^{-1}} & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \\
\hat{V}(\overline{\mathcal{M}}_{r'}(S)^N_{\leq \kappa_0 c/N})^{T^{-1}} & \longrightarrow & \hat{V}(\overline{\mathcal{M}}_{r'}(S)^{2N}_{\leq \kappa_1 c/N})^{T^{-1}} & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \\
\vdots & & \vdots & & \dots
\end{array}$$

10.6. The goal is to prove, by induction, that the first row of this system of double complexes is weakly exact in degrees $\leq m$, for suitable constants. The strategy, is to apply a quantitative, normed, spectral sequence argument. See [2, Prop. 9.6] for the precise statement.

This argument requires three assumptions:

(*i*) Rows 0 up to $m+1$ must be weakly exact in degrees $\leq m-1$, again for suitable constants.

(*ii*) Columns 0 up to m are weakly exact in degrees $\leq m$, for suitable constants.

(*iii*) The chain map between rows 0 and 1 must be homotopic to a chain map of small norm.

The first condition follows from the induction hypothesis. This induction hypothesis is necessarily quite subtle, because every row must be of such a shape that one can again apply to it:

(*i*) the construction of the system of double complexes,

(*ii*) which should then satisfy the three assumptions above.

Making sure that the second assumption can be applied to the double complexes constructed from the higher rows is a delicate step.

10.7. Verifying the second assumption above, is relatively straight-forward, modulo a crucial combinatorial input explained in the next paragraph.

The outline of the argument is as follows: Let $X_\bullet \to Y$ be the Čech cover of a *surjective* map $X \to Y$. Now apply $\hat{V}(\_)$ and take alternating sums of the face maps. Then the result is an exact complex, for which one has a precise control over the norms of the differentials.

We are precisely in the setting where Lemma 9.9 applies. In a final step, one passes from $\hat{V}(\_)$ to $\hat{V}(\_)^{T^{-1}}$, but we omit the details of this computation.

10.8. The tricky part in applying the argument outlined in the preceding paragraph is the surjectivity assumption.

Due to the omitted details, it might seem as if we need to prove that $\sum\colon M^N_{\leq c/N} \to M_{\leq c}$ is surjective. However, it is sufficient to show something slightly weaker, namely that the map $\sum\colon M^N \to M$ allows one to split $x \in M_{\leq c}$ into an N-tuple $x_i \in M_{\leq c/N+\delta}$, for some well-controlled error term δ.

This is a non-trivial combinatorial result, because we are now working with integral coefficients, instead of working over the reals. In particular, because we need it not just for $\overline{\mathcal{M}}_{r'}(S)$, but also for all the fibered products that occur in the Čech cover, and iteratively for Čech covers of those, etc. See [2, Lem. 9.8] for details.

10.9. Finally, we need to verify the third assumption of the spectral sequence argument. By a property of Breen–Deligne resolutions, the map

$$\sum\nolimits^* \colon \hat{V}(M^\bullet_\bullet)^{T^{-1}} \to \hat{V}(M^{N\bullet}_{\bullet/N})^{T^{-1}}$$

is chain homotopic to the formal sum of the maps

$$\pi_i^* \colon \hat{V}(M^\bullet_\bullet)^{T^{-1}} \to \hat{V}(M^{N\bullet}_{\bullet/N})^{T^{-1}},$$

where $\pi_i\colon M^N \to M$ is the i-th projection.

By a careful computation, which once again relies on many omitted details, one can show that the norm of this formal sum can be made arbitrarily small by choosing N sufficiently large.

10.10. Expanding on this summary goes beyond the scope of this text. For details, see [2, §IX] and `https://leanprover-community.github.io/liquid/`.

11. Breen–Deligne resolutions

11.1. **Theorem** (Breen, Deligne, [1, Appendix to §IV]). *There exists a functorial resolution of an abelian group A of the form*

$$\cdots \to \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}] \to \cdots \to \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0$$

where all n_i and $r_{i,j}$ are natural numbers.

11.2. The existence of Breen–Deligne resolutions is highly non-constructive. As a consequence, it is hard to compute with them, unless the goal is a vanishing result.

We are interested in the vanishing of certain Ext-groups, and for this purpose the following related result suffices.

11.3. **Theorem.** *There exists a functorial complex $Q'(A)$ of an abelian group A of the form*

$$\cdots \to \mathbb{Z}[A^{2^i}] \to \ldots \mathbb{Z}[A^4] \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0,$$

with the following property: Let A and B be abelian groups. If for all i, we have $\mathrm{Ext}^i(Q'(A), B) = 0$, then also $\mathrm{Ext}^i(A, B) = 0$ for all i.

11.4. The complex $Q'(A)$ shows up in the literature on MacLane homology, although a certain normalized version $Q(A)$ seems to be more well-known.

Without going into details, we state two properties of Q':

- (i) For morphisms $f, g \colon A \to B$, there is a natural homotopy of chain maps between $Q'(f+g)$ and $Q'(f) + Q'(g)$.
- (ii) There is a natural isomorphism $Q'(A) \cong Q'(A)^{\oplus 2}[1] \oplus \mathbb{Z} \otimes_{\mathbb{S}} A$.

References

[1] P. Scholze. "Lectures on Condensed Mathematics". 2019.
[2] P. Scholze. "Lectures on Analytic Geometry". 2020.