Action of the Weyl group

As usual let $R \subset E$ be a root system, with Weyl group $\mathcal{W}$.

Theorem Let Δ be a base of R

(i) If $x \in E$ is regular, then there is a $\sigma \in \mathcal{W}$ such that $\sigma(x)$ is in the fundamental Weyl chamber of Δ.

(ii) If Δ' is another base, then there is a $\sigma \in \mathcal{W}$ with $\sigma(\Delta') = \Delta$.

(iii) If $\alpha \in R$, then there is a $\sigma \in \mathcal{W}$ with $\sigma(\alpha) \in \cancel{R}.\ \Delta$

(iv) $\mathcal{W}$ is generated by the σ_α for $\alpha \in \Delta$.

(v) If $\sigma \in \mathcal{W}$ fixes Δ, then $\sigma = 1$.

<u>Proof</u> Let W' be the subgroup generated by the σ_α for $\alpha \in \Delta$.
In (iv) we prove $W' = W$. We now prove (i) – (iii) for W'.

(i) Consider $\delta = \frac{1}{2} \sum_{\alpha > 0} \alpha$. Let $\sigma \in W'$ be such that $(\sigma(x), \delta)$ is as big as possible.

If $\alpha \in \Delta$, recall that $\sigma_\alpha(\delta) = \delta - \alpha$. Also, note that $\sigma_\alpha \sigma \in W'$.

Hence $(\sigma(x), \delta) \geq (\sigma_\alpha \sigma(x), \delta) = (\sigma(x), \sigma_\alpha(\delta)) = (\sigma(x), \delta) - (\sigma(x), \alpha)$.

Therefore $(\sigma(x), \alpha) \geq 0$. Since x is regular $\sigma(x, \alpha) \neq 0$.

This means that $\sigma(x)$ is in the fundamental Weyl chamber.

(ii) We just showed that W' permutes Weyl chambers. Hence it also permutes bases.

(iii) By (ii) it suffices to find some regular $x \in E$ such that $\alpha \in \Delta(x)$.

Choose x close to P_α in such a way that

$$(x, \alpha) = \varepsilon > 0 \quad \text{and} \quad |(x, \beta)| > \varepsilon \quad \text{for all } \beta \neq \pm\alpha$$

Then α is a simple positive root in $\Delta(x)$.

Exercise: check in low rank examples that such an x exists.

(iv) It suffices to show that $\sigma_\alpha \in W'$ for $\alpha \in R$.

Using (iii) there is a σ such that $\sigma(\alpha) \in \Delta$. Hence $\sigma_{\sigma(\alpha)} \in W'$.

Now we use again the fact that $\sigma_\alpha = \sigma^{-1} \sigma_{\sigma(\alpha)} \sigma$ to deduce $\sigma_\alpha \in W'$.

(v) Assume $\sigma = \sigma_1 \cdots \sigma_t$ with $\sigma_i = \sigma_{\alpha_i}$, t minimal, and $\sigma(\Delta) = \Delta$.

Last time we saw that this implies $\sigma(\alpha_t) < 0$ which contradicts $\sigma(\Delta) = \Delta$.

Hence σ does not admit an expression $\sigma = \sigma_1 \cdots \sigma_t$, so $\sigma = 1$. ∎

In the final step of the proof, we used once again expressions of the form

$$\sigma = \sigma_1 \cdots \sigma_t$$

with t minimal. Such expressions are called reduced and $\ell(\sigma) = t$ is the length of σ. By definition $\ell(1) = 0$.

Let $n(\sigma)$ be the number of positive roots for which $\sigma(\alpha) < 0$.

<u>Lemma</u> $\ell(\sigma) = n(\sigma)$

<u>Proof</u> By induction on the length of σ. If $\ell(s) = 0$, then $\sigma = 1$ and the claim is true. Now assume the statement holds for all τ with $\ell(\tau) < \ell(\sigma)$.

Write σ in reduced form $\sigma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$. Put $\alpha = \alpha_t$.

Last time we saw $\sigma(\alpha) < 0$. We also saw that σ_α permutes all positive roots apart from α. Hence $n(\sigma\sigma_\alpha) = n(\sigma) - 1$.

But $\ell(\sigma\sigma_\alpha) = \ell(\sigma_{\alpha_1} \cdots \sigma_{\alpha_{t-1}}) = \ell(\sigma) - 1$. We are done by induction. ∎

We have seen that every element of E can be mapped into the closure of the fundamental Weyl chamber $\overline{\mathcal{C}(\Delta)}$

We will now see that $\overline{\mathcal{C}(\Delta)}$ is a fundamental domain for the action of $\mathcal{W}$.

Lemma Let $x, y \in \overline{\mathcal{C}(\Delta)}$. If $\sigma(x) = y$ for some $\sigma \in W$, then σ is a product of simple reflections σ_α $(\alpha \in \Delta)$ that fix x.
In particular $x = y$.

Proof If $\ell(\sigma) = 0$ then this is clear. Now suppose $\ell(\sigma) > 0$.

By the previous lemma, σ must send some positive root to a negative root,

and in particular there must be some simple root α with $\sigma(\alpha) < 0$.

Hence $0 \geq (\mu, \sigma\alpha) = (\sigma^{-1}\mu, \alpha) = (\lambda, \alpha) \geq 0$, since $\lambda, \mu \in \overline{\mathcal{C}(\Delta)}$.

We find $(\lambda, \alpha) = 0$, hence $\sigma_\alpha \lambda = \lambda$, thus $\sigma\sigma_\alpha \lambda = \mu$. Since $\ell(\sigma\sigma_\alpha) < \ell(\sigma)$,

we are done by induction $\blacksquare$