Towards classifying representations: the weight lattice

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $K = \overline{K}$, $\mathrm{char}(K) = 0$.
Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and let (R, E) be the root system corresponding to $\mathfrak{h} \subset \mathfrak{g}$.

Let V be a finite-dimensional representation of $\mathfrak{g}$.
As we have seen several times, we can decompose

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda \qquad \lambda \text{ is a weight of } V_\lambda \text{ if } V \neq 0.$$

Goal of this lecture: abstract theory of weights.

Let $\Lambda_R \subset E$ be the root lattice: the subgroup generated by R.
Clearly, Λ_R is a free abelian group of rank $\ell = \dim(E)$.
If Δ is a base of R, then Δ is a basis of Λ_R.

Let Λ_W be the set $\{ \lambda \in E \mid \langle \lambda, \alpha \rangle \in \mathbb{Z} \text{ for all } \alpha \in R \}$
Note that $\langle \lambda, \alpha \rangle = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}$ is linear in λ, and therefore
Λ_W is a subgroup of E. It is called the weight lattice.
Elements $\lambda \in \Lambda_W$ are called weights.

Observation: $\Lambda_R \subset \Lambda_W$.

Fix a base $\Delta \subset R$.

A weight $\lambda \in \Lambda_W$ is dominant if $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Delta$, and strongly dominant if $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Delta$.

We denote by Λ_W^+ the subset of dominant weights.

Note that $\Lambda_W^+ = \Lambda_W \cap \overline{\mathcal{C}(\Delta)}$ is exactly the set of all weights that lie in the closure of the fundamental Weyl chamber relative to Δ.

Similarly, $\Lambda_W \cap \mathcal{C}(\Delta)$ is the set of strongly dominant weights.

Exercise: draw Λ_W in our pictures of $A_1 \times A_1$, A_2, A_3, B_2, G_2.

Suppose $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$. Consider the weights $\lambda_1, \ldots, \lambda_\ell$ given by:

$$\frac{2(\lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}$$

These weights are dominant by construction, and they are the dual basis of the basis $\left\{ \frac{2\alpha_i}{(\alpha_i, \alpha_i)} \;\middle|\; i = 1, \ldots, \ell \right\}$.

The weights $\lambda_1, \ldots, \lambda_\ell$ are called the fundamental dominant weights relative to Δ, and they form a basis for Λ_W.

Exercise: draw the fundamental dominant weights in our pictures of $A_1 \times A_1$, A_2, A_3, B_2, G_2.

Fundamental group

Since $\Lambda_R \subset \Lambda_W$ are both lattices of rank ℓ,

the quotient Λ_W / Λ_R must be a finite group.

It is the fundamental group of R.

The cardinality of Λ_W / Λ_R is the determinant of the base change matrix

from $\{\alpha_1, \dots, \alpha_\ell\}$ to $\{\lambda_1, \dots, \lambda_\ell\}$.

Write $\alpha_i = \sum' m_{ij} \lambda_j$. Then $(m_{ij})_{ij}$ is a matrix of integers, and

$$m_{ij} = \sum_k m_{ik} \delta_{kj} = \sum_k m_{ik} \langle \lambda_k, \alpha_j \rangle = \langle \alpha_i, \alpha_j \rangle.$$

So this is none other than the Cartan matrix!

<u>Exercise</u> Compute the determinants of Cartan matrices of irreducible root systems.

Saturated sets of weights

Our long term goal is to understand the sets of weights that arise from representations of Lie algebras. (Remember what we did for $\mathfrak{sl}_3$.)

With this in mind, we make the following definition.

A subset $P \subset \Lambda_W$ is saturated if for all $\lambda \in P$ and $\alpha \in R$, and all i between 0 and $\langle \lambda, \alpha \rangle$ the weight $\lambda - i\alpha$ is in P.

Example: R itself is a saturated set.

A saturated set P has highest weight λ if $\lambda \in X$ and for all $\mu \in X$ we have $\mu \leq \lambda$.

(Recall $\mu \leq \lambda$ if $\lambda - \mu$ is an $\mathbb{N}$-linear sum of positive roots.)

Our goal for today:

Suppose that P is saturated with highest weight λ.
Then P consists of all dominant weights μ with $\mu \leq \lambda$
and all their $\mathcal{W}$-conjugates. It is a finite set.

The Weyl group $\mathcal{W}$ preserves the inner product on E, and hence preserves Λ_W.

More precisely, check that $\sigma_{\alpha_i} \lambda_j = \lambda_j - \delta_{ij} \alpha_i$.

Recall that the closure of the fundamental Weyl chamber $\overline{\mathcal{C}(\Delta)}$ is a fundamental domain for the action of $\mathcal{W}$ on E.

Hence we get:

<u>Lemma</u> Every weight is conjugate under $\mathcal{W}$ to exactly one dominant weight.
If λ is dominant, then $\sigma\lambda \leq \lambda$ for all $\sigma \in \mathcal{W}$.
If λ is strongly dominant, then $\sigma\lambda = \lambda$ only if $\sigma = \mathrm{id}$. $\blacksquare$

<u>Lemma</u> For $\lambda \in \Lambda_W^+$, the number of dominant weights $\mu \leq \lambda$ is finite.

<u>Proof</u> Suppose that $\mu \in \Lambda_W^+$ and $\mu \leq \lambda$. Then $\lambda + \mu \in \Lambda_W^+$,

and $\lambda - \mu$ is a sum of positive roots. Hence $(\lambda+\mu, \lambda-\mu) \geq 0$.

But $(\lambda+\mu, \lambda-\mu) = (\lambda,\lambda) - (\mu,\mu)$.

So μ lies in the compact set $\{ x \in E \mid (x,x) \leq (\lambda,\lambda) \}$.

But μ also lies in the discrete set Λ_W^+.

The intersection

$$\{ x \in E \mid (x,x) \leq (\lambda,\lambda) \} \cap \Lambda_W^+$$

is finite. $\blacksquare$

Recall $\delta = \frac{1}{2} \sum_{\alpha \in R^+}' \alpha$. We have seen $\sigma_{\alpha_i} \delta = \delta - \alpha_i$, for $\alpha_i \in \Delta$.

Note that δ is not always a root. However:

<u>Lemma</u> $\delta = \sum_{i=1}^{\ell}{}' \lambda_i$ and hence a strongly dominant weight.

<u>Proof</u> We have $\delta - \alpha_i = \sigma_{\alpha_i} \delta = \delta - \langle \delta, \alpha_i \rangle \alpha_i$. Hence $\langle \delta, \alpha_i \rangle = 1$ for all $\alpha_i \in \Delta$. Hence $\delta = \sum' \langle \delta, \alpha_i \rangle \lambda_i = \sum \lambda_i$. ∎

Describing saturated sets

Let P be a saturated set. Since $\sigma_\alpha \lambda = \lambda - \langle \lambda, \alpha \rangle \alpha$, we see from the definition of saturated sets that they must be stable under W.

Corollary If P has highest weight λ, then P is finite.

Proof There are only finitely many dominant weight $\leq \lambda$, and P is W-stable.

Finally, we prove a lemma that shows that saturated sets are fully determined by their highest weight (if they have one).

<u>Lemma</u> Let P be saturated, with highest weight λ.

If $\mu \in \Lambda_V^+$ and $\mu \leq \lambda$, then $\mu \in P$.

<u>Proof</u> Suppose that $\mu' = \mu + \sum_{\alpha \in \Delta} k_\alpha \alpha$, $k_\alpha \in \mathbb{N}$ and $\mu' \in P$.

An example of such a μ' is λ.

Suppose that $\mu' \neq \mu$, so that $\sum' k_\alpha > 0$. From $\left(\sum k_\alpha \alpha, \sum k_\alpha \alpha\right) > 0$

we deduce $\left(\sum' k_\alpha \alpha, \beta\right) > 0$ for some $\beta \in \Delta$, with $k_\beta > 0$.

Hence $\langle \sum' k_\alpha \alpha, \beta \rangle > 0$. Since μ is dominant, we have $\langle \mu, \beta \rangle \geq 0$,

and thus $\langle \mu', \beta \rangle > 0$. Hence 1 lies between 0 and $\langle \mu, \beta \rangle$.

By definition of saturated set, we find $\mu' - \beta \in P$.

Repeatedly applying this argument, we conclude that $\mu \in P$. $\blacksquare$

Lemma Let $\mu \in \Lambda_W^+$, $\nu = \sigma\mu$ for some $\sigma \in W$. Then $(\nu+\delta, \nu+\delta) \leq (\mu+\delta, \mu+\delta)$ with equality only if $\mu = \nu$.

Proof We compute

$$(\nu+\delta, \nu+\delta) = (\sigma^{-1}(\nu+\delta), \sigma^{-1}(\nu+\delta))$$
$$= (\mu+\sigma^{-1}\delta, \mu+\sigma^{-1}\delta)$$
$$= (\mu+\delta, \mu+\delta) - 2(\mu, \delta - \sigma^{-1}\delta)$$

Now $\mu \in \Lambda_W^+$ by assumption, and $\sigma^{-1}\delta \leq \delta$ since δ is dominant.
Hence $(\mu, \delta - \sigma^{-1}\delta) \geq 0$, with equality only if

$$(\mu, \delta) = (\mu, \sigma^{-1}\delta) = (\sigma\mu, \delta) = (\nu, \delta).$$

In other words, if $(\mu - \nu, \delta) = 0$. But $\nu = \sigma\mu \leq \mu$ since μ is dominant.
Hence $\mu - \nu$ is a sum of positive roots, and δ is strongly dominant, so $\mu = \nu$.

<u>Corollary</u> Let P be a saturated set, with highest weight λ.

If $\mu \in P$, then $(\mu+\delta, \mu+\delta) \leq (\lambda+\delta, \lambda+\delta)$ with equality only if $\mu = \lambda$.

<u>Proof</u> By the lemma, it suffices to prove this if μ is dominant.

Note that $\pi = \lambda - \mu$ is a sum of positive roots. Now use $\pi = (\lambda+\delta) - (\mu - \delta)$

to calculate

$$(\lambda+\delta, \lambda+\delta) - (\mu+\delta, \mu+\delta) = (\lambda+\delta, \pi) - (\pi, \mu+\delta) \geq 0$$

The final inequality holds because $\lambda+\delta$ and $\mu+\delta$ are dominant.

Since they are strongly dominant, equality holds only if $\pi = 0$. ∎