The universal enveloping algebra

Goal: Let $\mathfrak{g}$ be a Lie algebra.

We will construct an associative algebra $U(\mathfrak{g})$,

the universal enveloping algebra, with a homomorphism $\mathfrak{g} \xrightarrow{i} U(\mathfrak{g})$

such that for every associative algebra A,

commutator bracket

and homomorphism of Lie algebras $\varphi : \mathfrak{g} \longrightarrow A$

there is a unique homomorphism

$\tilde{\varphi} : U(\mathfrak{g}) \longrightarrow A$ such that

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "i"] \arrow[dr, "\varphi"'] & U(\mathfrak{g}) \arrow[d, "\tilde{\varphi}"] \\ & A \end{tikzcd}$$

commutes.

Motivation The universal enveloping algebra will be a useful tool in constructing representations of $\mathfrak{g}$.
Indeed, there is a dictionary (equivalence of categories) between representations of $\mathfrak{g}$ and modules over $U(\mathfrak{g})$.

Uniqueness The universal enveloping algebra is characterised by its universal property: If $\mathfrak{g} \xrightarrow{i'} U'(\mathfrak{g})$ is another algebra with the same property, then i and i' extend to homomorphisms

$$\begin{tikzcd} U(\mathfrak{g}) \arrow[rr, bend left, "\tilde{i}"] & & U'(\mathfrak{g}) \arrow[ll, bend left, "\tilde{i}'"] \\ & \mathfrak{g} \arrow[ul, "i"] \arrow[ur, "i'"'] & \end{tikzcd}.$$

Now observe that $\tilde{i} \circ \tilde{i}'$ and $\tilde{i}' \circ \tilde{i}$ extend the identity on $\mathfrak{g}$ and therefore must be identity maps themselves. Thus the establish a canonical isomorphism $U(\mathfrak{g}) \cong U'(\mathfrak{g})$.

<u>Construction</u> Recall the tensor algebra $T(\mathfrak{g}) = \bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n}$ that we saw earlier in the course.

Let $\mathfrak{J}$ be the ideal of $T(\mathfrak{g})$ generated by

$$\{ x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g} \}.$$

We define $\mathcal{U}(\mathfrak{g})$ as $T(\mathfrak{g})/\mathfrak{J}$, and the map $i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is the composition $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^{\otimes 1} \hookrightarrow T(\mathfrak{g}) \xrightarrow{q} T(\mathfrak{g})/\mathfrak{J} = \mathcal{U}(\mathfrak{g})$.

where q is the quotient map. Clearly

$$i([x, y]) = q([x, y]) = q(x \otimes y - y \otimes x) = i(x)i(y) - i(y)i(x) = [i(x), i(y)].$$

Next, we verify the universal property.

Let $\varphi : \mathfrak{g} \to A$ be a homomorphism of Lie algebras, where A is an associative algebra with the commutator bracket.

In particular, φ is linear, so we get an induced morphism

$$T(\mathfrak{g}) \xrightarrow{\varphi'} A$$

Since φ preserves the Lie bracket, J lies in the kernel of φ'.
Therefore we get a morphism

$$\tilde{\varphi} : U(\mathfrak{g}) = T(\mathfrak{g})/J \to A$$

as desired.