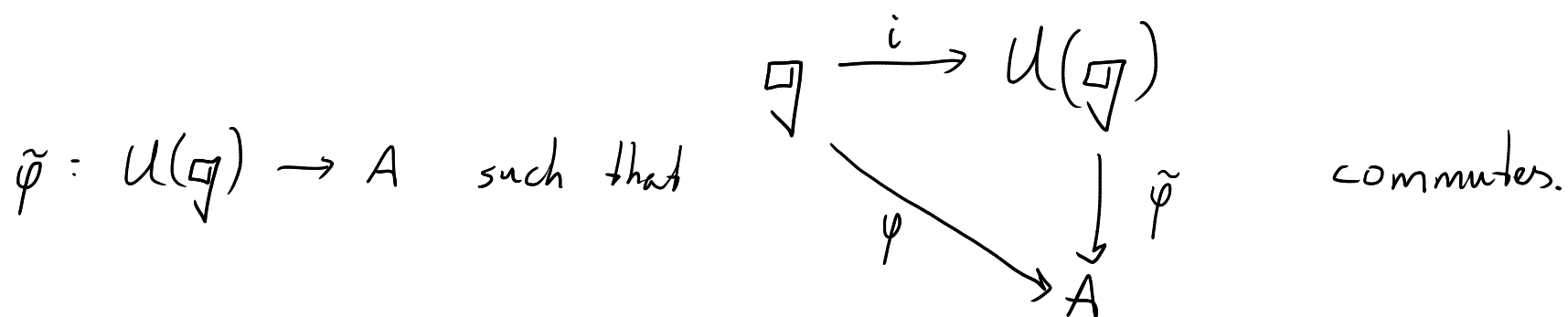


The universal enveloping algebra

Goal: Let \mathfrak{g} be a Lie algebra.

We will construct an **associative algebra** $U(\mathfrak{g})$,

the **universal enveloping algebra**, with a homomorphism $\mathfrak{g} \xrightarrow{i} U(\mathfrak{g})$
such that for every associative algebra A ,
and homomorphism of Lie algebras $\varphi: \mathfrak{g} \rightarrow A$ \leftarrow commutator bracket
there is a **unique** homomorphism



Motivation The universal enveloping algebra will be a useful tool in constructing representations of \mathfrak{g} .

Indeed, there is a dictionary (equivalence of categories) between representations of \mathfrak{g} and modules over $U(\mathfrak{g})$.

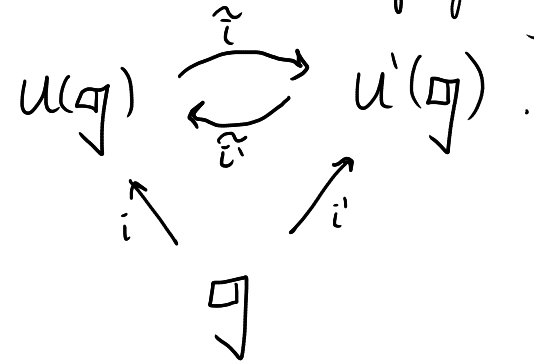
Uniqueness The universal enveloping algebra is characterised by its universal property: If $\mathfrak{g} \xrightarrow{i} U(\mathfrak{g})$ is another algebra with the same property,

then i and i' extend to homomorphisms $U(\mathfrak{g}) \xrightarrow{\tilde{i}} U'(\mathfrak{g})$.

Now observe that $\tilde{i} \circ i$ and $i' \circ i$ extend

the identity on \mathfrak{g} and therefore must be

identity maps themselves. Thus we establish a canonical isomorphism $U(\mathfrak{g}) \cong U'(\mathfrak{g})$.



Construction Recall the **tensor algebra** $T(\mathfrak{g}) = \bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n}$

that we saw earlier in the course.

Let \mathcal{I} be the ideal of $T(\mathfrak{g})$ generated by

$$\{x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g}\}.$$

We define $U(\mathfrak{g})$ as $T(\mathfrak{g})/\mathcal{I}$, and the map $i: \mathfrak{g} \rightarrow U(\mathfrak{g})$

is the composition $\mathfrak{g} \rightarrow \mathfrak{g}^{\otimes 1} \hookrightarrow T(\mathfrak{g}) \xrightarrow{q} T(\mathfrak{g})/\mathcal{I} = U(\mathfrak{g})$.

where q is the quotient map. Clearly

$$i([x, y]) = q([x, y]) = q(x \otimes y - y \otimes x) = i(x)i(y) - i(y)i(x) = [i(x), i(y)].$$

Next, we verify the universal property.

Let $\varphi: \mathfrak{g} \rightarrow A$ be a homomorphism of Lie algebras, where A is an associative algebra with the commutator bracket.

In particular, φ is linear, so we get an induced morphism

$$T(\mathfrak{g}) \xrightarrow{\varphi'} A$$

Since φ preserves the Lie bracket, \mathcal{J} lies in the kernel of φ' .

Therefore we get a morphism

$$\tilde{\varphi}: U(\mathfrak{g}) = T(\mathfrak{g})/\mathcal{J} \rightarrow A$$

as desired.