Towards Root Spaces

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over an algebraically closed field K of characteristic 0.

From last time: there exists a Cartan subalgebra

$$\mathfrak{h} \subset \mathfrak{g}$$

Fix such a subalgebra.

Decompositions

We have seen these several times by now:

If V is a finite-dimensional representation, then

$$V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$$

weight space

weight

where $\boxed{X(v) = \alpha(X) \cdot v}$ for all $X \in \mathfrak{h}$ and $v \in V_\alpha$.

In particular we can apply this to the adjoint representation.

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$$

- $\mathfrak{g}_0 = \mathfrak{h}$
- If $\mathfrak{g}_\alpha \neq 0$ and $\alpha \neq 0$ then:
  - α is a root
  - $\mathfrak{g}_\alpha$ is a root space

Lemma

(i) For all $\alpha, \beta \in \mathfrak{h}^*$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

(ii) If $x \in \mathfrak{g}_\alpha$ and $\alpha \neq 0$, then $\mathrm{ad}\, x$ is nilpotent.

(iii) If $\alpha, \beta \in \mathfrak{h}^*$, and $\alpha + \beta \neq 0$, then $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{g}_\beta$ relative to the Killing form of $\mathfrak{g}$.

Proof (i) This is the usual computation, see the proofs for $\mathfrak{sl}_2$ or $\mathfrak{sl}_3$.

<u>Proof (cntd)</u> (ii) follows from (i): $(\operatorname{ad} x)^n$ will map $\mathfrak{g}_\beta$ to $\mathfrak{g}_{\beta+n\alpha}$. Since $\mathfrak{g}$ is finite-dimensional, there is an n such that $\mathfrak{g}_{\beta+n\alpha} = 0$ for all roots β.

(iii) Let $H \in \mathfrak{h}$ be such that $(\alpha+\beta)(H) \neq 0$.

For all $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$ we find

$$B([H,X],Y) = -B([X,H],Y) = -B(X,[H,Y]).$$

In other words $\alpha(H)\,B(X,Y) = -\beta(H)\,B(X,Y)$.

Since $(\alpha+\beta)(H) \neq 0$, this forces $B(X,Y) = 0$. $\square$

Let $R = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0 \text{ and } \alpha \neq 0\}$ be the roots of $\mathfrak{g}$.

Since $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\alpha + \beta \neq 0$ and B is nondegenerate,

we find that $B|_{\mathfrak{h}}$ is nondegenerate, and hence:

> For every $\alpha \in \mathfrak{h}^*$, there is a unique $t_\alpha \in \mathfrak{h}$
> such that $\alpha(H) = B(t_\alpha, H)$ for all $H \in \mathfrak{h}$.

Proposition (i) R spans $\mathfrak{h}^*$.

(ii) If $\alpha \in R$, then $-\alpha \in R$.

(iii) Let $\alpha \in R$, $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$. Then $[X,Y] = B(X,Y) \cdot t_\alpha \in \mathfrak{h}$.

(iv) If $\alpha \in R$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is 1-dimensional, with basis t_α.

(v) $\alpha(t_\alpha) = B(t_\alpha, t_\alpha) \neq 0$ for $\alpha \in R$.

(vi) If $\alpha \in R$ and $X_\alpha \in \mathfrak{g}_\alpha$ nonzero, then there exists $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that with $H_\alpha = [X_\alpha, Y_\alpha]$, $(X_\alpha, H_\alpha, Y_\alpha)$ is an $\mathfrak{sl}_2$-triple: they span a 3-dimensional subalgebra isomorphic to $\mathfrak{sl}_2$ via:

$$X_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y_\alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

(vi) $H_\alpha = \dfrac{2t_\alpha}{B(t_\alpha, t_\alpha)}$ and $H_\alpha = -H_{-\alpha}$.

Proof (i) If R does not span $\mathfrak{h}^*$, then there exists a nonzero $H \in \mathfrak{h}$ such that $\alpha(H) = 0$ for all $\alpha \in R$. This means that $[H, \mathfrak{g}_\alpha] = 0$ for all $\alpha \in R$. But $\mathfrak{h}$ is abelian, so we also have $[H, \mathfrak{h}] = 0$ and hence $[H, \mathfrak{g}] = 0$. This means that H generates an non-trivial abelian ideal of $\mathfrak{g}$. But $\mathfrak{g}$ is semisimple. $\lightning$

(ii) Since B is nondegenerate $B(\mathfrak{g}_\alpha, \mathfrak{g}) \neq 0$.

But for $\beta \neq -\alpha$ we have $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ by the lemma.
Hence $\mathfrak{g}_{-\alpha} \neq 0$, and $-\alpha \in R$.

(iii) Let $\alpha \in R$, $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$. Then $[X,Y] = B(X,Y) \cdot t_\alpha \in \mathfrak{h}$.

Pick some $H \in \mathfrak{h}$. Then we get

$$B(H, [X,Y]) = B([H,X], Y) = \alpha(H) B(X,Y) = B(t_\alpha, H) B(X,Y)$$

$$= B(B(X,Y) t_\alpha, H) = B(H, B(X,Y) t_\alpha).$$

Hence $\mathfrak{h}$ is orthogonal to $[X,Y] - B(X,Y) t_\alpha$.

But $[X,Y] - B(X,Y) t_\alpha \in \mathfrak{h}$ and must hence be 0.

We conclude $[X,Y] = B(X,Y) t_\alpha$.

(iv) If $\alpha \in R$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is 1-dimensional, with basis t_α.

By part (iii) we know that t_α spans $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$.

We need to show it is nonzero. Pick some nonzero $X \in \mathfrak{g}_\alpha$.

If $B(X, \mathfrak{g}_{-\alpha}) = 0$, then $B(X, \mathfrak{g}) = 0$ which is impossible,

since B is nondegenerate.

Hence there exists a $Y \in \mathfrak{g}_{-\alpha}$ such that $B(X, Y) \neq 0$

By (iii), we find $[X, Y] \neq 0$.

(v) $\alpha(t_\alpha) = B(t_\alpha, t_\alpha) \neq 0$ for $\alpha \in R$.

Suppose that $\alpha(t_\alpha) = 0$. Then $[t_\alpha, X] = 0 = [t_\alpha, Y]$ for all $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$.

As before, we can find such X and Y with $B(X, Y) \neq 0$.

By scaling Y, we can assume $B(X, Y) = 1$, so that $[X, Y] = t_\alpha$.

Hence X, t_α, and Y span a 3-dimensional subalgebra

that is solvable. Hence $\mathrm{ad}(t_\alpha)$ is nilpotent, but it is also

semisimple since t_α is contained in the Cartan subalgebra $\mathfrak{h}$.

Hence $\mathrm{ad}(t_\alpha) = 0$ and thus $t_\alpha = 0$. This contradicts (iv), so $\alpha(t_\alpha) \neq 0$.

(vi) If $\alpha \in R$ and $X_\alpha \in \mathfrak{g}_\alpha$ nonzero, then there exists $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that with $H_\alpha = [X_\alpha, Y_\alpha]$, $(X_\alpha, H_\alpha, Y_\alpha)$ is an $\mathfrak{sl}_2$-triple

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Given $X_\alpha \in \mathfrak{g}_\alpha$ nonzero, pick $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that

$$B(X_\alpha, Y_\alpha) = \frac{2}{B(t_\alpha, t_\alpha)} = \frac{2}{\alpha(t_\alpha)}$$

Now put $H_\alpha = B(X_\alpha, Y_\alpha) \cdot t_\alpha$, so that $H_\alpha = [X_\alpha, Y_\alpha]$.

Also, $[H_\alpha, X_\alpha] = \frac{2}{\alpha(t_\alpha)} [t_\alpha, X_\alpha] = \frac{2}{\alpha(t_\alpha)} \cdot \alpha(t_\alpha) \cdot X_\alpha = 2X_\alpha$.

Similarly $[H_\alpha, Y_\alpha] = -2Y_\alpha$. We know recognise

$(X_\alpha, H_\alpha, Y_\alpha)$ as an $\mathfrak{sl}_2$-triple.

(vi) $H_\alpha = \dfrac{2t_\alpha}{B(t_\alpha, t_\alpha)}$ and $H_\alpha = -H_{-\alpha}$.

The first part is clear from the computations above.

In particular H_α does not depend on X_α.

Recall that $t_{-\alpha}$ is characterised by

$$B(t_{-\alpha}, H) = -\alpha(H) \quad \text{for all } H \in \mathfrak{h}.$$

Hence $B(t_{-\alpha}, H) = -B(t_\alpha, H)$ for all H, and we conclude

$t_{-\alpha} = -t_\alpha$, and thus $H_\alpha = -H_{-\alpha}$. $\square$

Proposition (i) For all $\alpha \in R$, we have $\dim \mathfrak{g}_\alpha = 1$.

In particular $S_\alpha = \mathfrak{g}_\alpha \oplus \langle H_\alpha \rangle \oplus \mathfrak{g}_{-\alpha}$ is 3-dimensional and $\cong \mathfrak{sl}_2$.

For any nonzero $X \in \mathfrak{g}_\alpha$, there exists a unique $Y \in \mathfrak{g}_{-\alpha}$ with $[X, Y] = H_\alpha$.

(ii) If $\alpha \in R$, then $\pm\alpha$ are the only scalar multiples of α that are roots.

(iii) If $\alpha, \beta \in R$, then $\beta(H_\alpha) \in \mathbb{Z}$, and $\beta - \beta(H_\alpha)\alpha \in R$.

(iv) If $\alpha, \beta \in R$, and $\alpha + \beta \in R$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

(v) Let $\alpha, \beta \in R$, with $\beta \neq \pm\alpha$. Let r, q be the largest integers for which $\beta - r\alpha$, $\beta + q\alpha$ are roots. Then $\beta + i\alpha \in R$ for all $-r \leq i \leq q$ and $\beta(H_\alpha) = r - q$.

(vi) $\mathfrak{g}$ is generated as Lie algebra by the root spaces $\mathfrak{g}_\alpha$, $\alpha \in R$.

all root spaces whose root is a multiple of α.

<u>Proof</u> Fix $\alpha \in R$. Consider $V = \mathfrak{h} \oplus \bigoplus_{c \in K^*} \mathfrak{g}_{c\alpha}$.

By the preceding proposition, we know that V is an S_α - subrepresentation of $\mathfrak{g}$.

So we can apply our knowledge of the representation theory of sl_2.

The element $H_\alpha \in S_\alpha$ acts on V via the weights

- 0 on $\mathfrak{h}$
- $2c$ on $\mathfrak{g}_{c\alpha}$ $\qquad [2c = c\alpha(H_\alpha)]$

And by the representation theory of sl_2, these weights must be integers.

This means that all c that occur, must lie in $\frac{1}{2} \cdot \mathbb{Z}$.

Now consider the weight space $V_0 = \mathfrak{h}$.

The root $\alpha \in \mathfrak{h}^*$ is a linear map $\mathfrak{h} \to K$, and $\mathrm{Ker}(\alpha)$ is a subrepresentation of V. Note that $\mathrm{Ker}(\alpha)$ has codimension 1 in V_0.

On the other hand $S_\alpha \subset V$ is also an irreducible subrepresentation.

Together, $\mathfrak{h} \oplus (S_\alpha \cap V_0)$ cover all of V_0, and so there can be no other subrepresentations of V with even weights, because they would contribute to V_0.

This shows that if α is a root, then 2α cannot be a root.

But then $\frac{1}{2}\alpha$ cannot be a root either. We conclude $V = \mathfrak{h} \oplus S_\alpha$.

In particular $\dim \mathfrak{g}_\alpha = 1$, and the only multiples of α that are roots are $\pm\alpha$. This proves (i) and (ii).

Now fix some $\beta \in R$, $\beta \neq \pm\alpha$. Consider $V = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\beta + i\alpha}$.

Again, this is an S_α-subrepresentation of $\mathfrak{g}$.

Note that H_α acts on $\mathfrak{g}_{\beta+i\alpha}$ with weight $\beta(H_\alpha) + 2i$.

These weights are all odd or all even integers.

In particular $\beta(H_\alpha) \in \mathbb{Z}$. Furthermore, the $\mathfrak{g}_{\beta+i\alpha}$ are 1-dimensional, and hence V is irreducible as representation of S_α.

Let q and r be the integers such that $\beta(H_\alpha)+2q$ (resp. $\beta(H_\alpha)-2r$) are the highest (resp. lowest) weights of V.

By the representation theory of $\mathfrak{sl}_2$ we find that

$$\beta+i\alpha \in R \iff -r \leq i \leq q.$$

We also know that $\beta(H_\alpha)-2r = -(\beta(H)_\alpha + 2q)$ by symmetry of the weights of V. Hence $\beta(H_\alpha) = r-q$.

Finally, if $\alpha+\beta \in R$, then $\mathrm{ad}\, \mathfrak{g}_\alpha$ maps $\mathfrak{g}_\beta$ onto $\mathfrak{g}_{\alpha+\beta}$.

Once again, this is because V is irreducible.

This finishes the proof of the remaining items.