Tensor products

Let $\mathfrak{g}$ be a Lie algebra, and let V and W be two representations of $\mathfrak{g}$.

The tensor product representation is defined via

$$X(v \otimes w) = X(v) \otimes w + v \otimes X(w)$$

(Note the similarity with the Leibniz rule.)

By induction, a representation on $V^{\otimes n} = \underbrace{V \otimes V \otimes \dots \otimes V}_{n \text{ times}}$

Symmetric powers and exterior powers

In a similar vein we get representations on

$\mathrm{Sym}^n V$ and $\Lambda^n V$:

$$X(v_1 \cdot v_2 \cdot \ldots \cdot v_n) = X(v_1) \cdot v_2 \cdot \ldots \cdot v_n + v_1 \cdot X(v_2) \cdot \ldots \cdot v_n + \ldots$$
$$+ v_1 \cdot v_2 \cdot \ldots \cdot X(v_n)$$

$$X(v_1 \wedge v_2 \wedge \ldots \wedge v_n) = X(v_1) \wedge \ldots \wedge v_n + \ldots + v_1 \wedge v_2 \wedge \ldots \wedge X(v_n).$$

Tensor products of representations of sl_2

Suppose that V and W are representations of sl_2.

Decompose them into eigenspaces for H:

$$V = \bigoplus V_\alpha \quad , \quad W = \bigoplus W_\beta$$

If $v \in V_\alpha$ and $w \in W_\beta$, then

$$\begin{aligned} H(v \otimes w) &= H(v) \otimes w + v \otimes H(w) \\ &= \alpha v \otimes w + v \otimes \beta w \\ &= \alpha (v \otimes w) + \beta (v \otimes w) \end{aligned}$$

Hence $v \otimes w \in (V \otimes W)_{\alpha+\beta}$

Let $V = \langle e_1, e_2 \rangle$ be the standard representation of SL_2,

so that $H(e_1) = e_1$ and $H(e_2) = -e_2$.

The element $(e_1)^n \in Sym^n(V)$ satisfies $H(e_1^n) = n \cdot H(e_1) \cdot e_1^{n-1}$

$$= n \cdot e_1^n$$

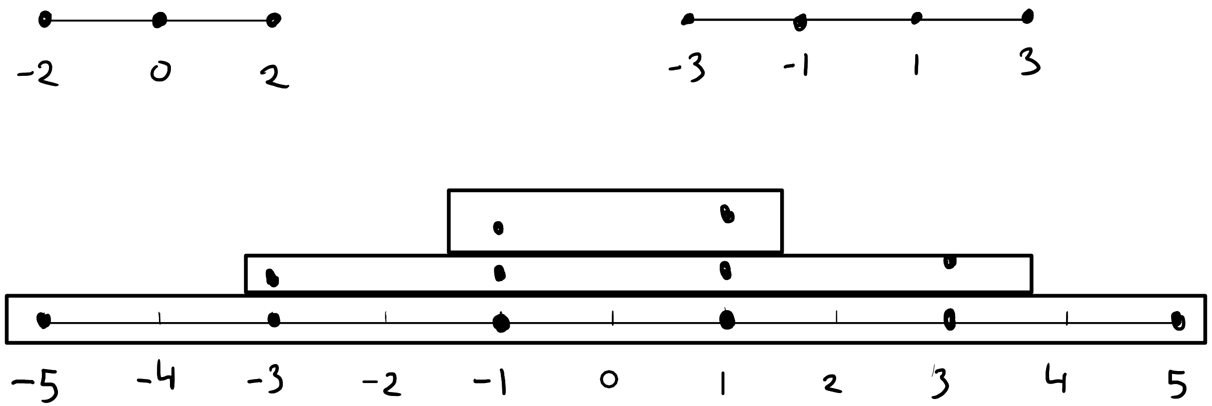
Since $\dim(Sym^n(V)) = \binom{n+1}{1}$, we find that $Sym^n(V)$

is the irreducible representation of dim $n+1$:

$$\begin{tikzcd} \underset{-n}{\bullet} \arrow[r, bend left, "X"] & \underset{-n+2}{\bullet} \arrow[l, bend left, "Y"] \arrow[r, bend left, "X"] & \cdots \cdots \arrow[r, bend left, "X"] & \underset{n-2}{\bullet} \arrow[r, bend left, "X"] & \underset{n}{\bullet} \arrow[l, bend left, "Y"] \end{tikzcd}$$

Can we understand $\mathrm{Sym}^2(V) \otimes \mathrm{Sym}^3(V)$?

We saw before that $(V_\alpha \otimes W_\beta) \subset (V \otimes W)_{\alpha+\beta}$.



Conclusion: $\mathrm{Sym}^2(V) \otimes \mathrm{Sym}^3(V) = \mathrm{Sym}^5(V) \oplus \mathrm{Sym}^3(V) \oplus V$.

<u>Exercise</u> Write $\mathrm{Sym}^m(V) \otimes \mathrm{Sym}^n(V)$ as sum of irreducibles.