Tensor products

Suppose that U, V, and W are vector spaces over some field K. Assume that

$$\dim U = k, \qquad \dim V = m, \qquad \dim W = n$$

What is the dimension of $\mathrm{Hom}(V, W)$?

After picking bases for V and W, it is the space of $n \times m$ - matrices. So, answer: $n \cdot m$

<u>Similar question</u>: what is the dimension of the vector space of bilinear maps $U \times V \to W$?

<u>Answer</u> A bilinear map $f : U \times V \to W$ gives a linear map $f(u, \_) : V \to W$ for every $u \in U$.

So it is the same as a linear map

$$U \to \mathrm{Hom}(V, W).$$

$\mathrm{Bil}(U \times V, W) \cong \mathrm{Hom}(U, \mathrm{Hom}(V, W))$. Dimension: $n \cdot m \cdot k$.

<u>Extension of scalars</u> Suppose that V is a vector space $/\mathbb{R}$, of dimension n.

Can we build a complex vector space of dimension n in a natural way, starting from V?

<u>Not so natural method</u> Pick a basis of V (over $\mathbb{R}$)

$$\{v_1, \dots, v_n\}$$

Now consider all $\mathbb{C}$-linear combinations of $v_1, v_2, \dots, v_n$.

Of course we can also think about multilinear maps

$$V_1 \times V_2 \times \dots \times V_r \longrightarrow W$$

It would be really useful if we can view this as a

linear map $\text{SomeConstruction}(V_1, V_2, \dots, V_r) \to W$,

because then we can apply all theorems of linear algebra to it.

This is exactly what tensor products will do for us.

$$\text{SomeConstruction}(U, V) = U \otimes V$$

Let's understand this better!

$\dim U = k$
$\dim V = m$
$\dim W = n$

What do we know already?

The dimension of $\mathrm{Bil}(U \times V, W)$ is $n \cdot m \cdot k$

The dimension of $\mathrm{Hom}(U \otimes V, W)$ is $n \cdot \dim(U \otimes V)$

So $\dim(U \otimes V)$ must be $m \cdot k$.

Suppose $\{u_1, \dots, u_k\}$ and $\{v_1, \dots, v_m\}$ are bases of U and V.

How do we get a vector space of dimension $k \cdot m$?

Generate one from $\{u_1, \dots, u_k\} \times \{v_1, \dots, v_m\}$. Works, but ugly :(

Building the tensor product without picking bases

Consider the vector space generated by the "formal" symbols:

$$u \otimes v \qquad \text{for all } u \in U \text{ and } v \in V.$$

Now take the quotient by the relations:

- $(\lambda u) \otimes v = \lambda (u \otimes v) = u \otimes (\lambda v)$
- $(u_1 + u_2) \otimes v = (u_1 \otimes v) + (u_2 \otimes v)$
- $u \otimes (v_1 + v_2) = (u \otimes v_1) + (u \otimes v_2)$

Linear maps out of tensor products

What do we need to do, to give a linear map $U \otimes V \to W$?

For all $u \in U$ and $v \in V$ give the image $f(u \otimes v)$

and make sure that $f((\lambda u) \otimes v) = f(\lambda(u \otimes v)) = f(u \otimes (\lambda v))$

and $f((u_1 + u_2) \otimes v) = f(u_1 \otimes v) + f(u_2 \otimes v)$

etc ....

But that is exactly a bilinear map $U \times V \to W$! :)

$$\mathrm{Hom}(U \otimes V, W) \cong \mathrm{Bil}(U \times V, W) \cong \mathrm{Hom}(U, \mathrm{Hom}(V, W))$$