Symmetric powers and exterior powers

We saw $\operatorname{Hom}(U \otimes V, W) = \operatorname{Bil}(U \times V, W)$

By a little variation we can also solve

$$\operatorname{Hom}(\text{SomeConstr.}(V, n), W) \cong \{\text{sym. multilinear maps } V^n \to W\}$$

and

$$\operatorname{Hom}(\text{SomeConstr.}(V, n), W) \cong \{\text{alt. multilinear maps } V^n \to W\}$$

These are the symmetric powers and exterior powers of V.

Notation: $\operatorname{Sym}^n V$ and $\Lambda^n V$

Constructions (symmetric power)

$\mathrm{Sym}^n V$ is the vector space generated by "formal" symbols

$$v_1 \cdot v_2 \cdot \ldots \cdot v_n \qquad \text{for all } (v_1, v_2, \ldots, v_n) \in V^n$$

modulo the relations

- $v_1 \cdot v_2 \cdot \ldots \cdot v_n = v_{\sigma(1)} \cdot v_{\sigma(2)} \cdot \ldots \cdot v_{\sigma(n)}$ for all permutations σ
- $\lambda (v_1 \cdot v_2 \cdot \ldots \cdot v_n) = (\lambda v_1) \cdot \ldots \cdot v_n = \ldots = v_1 \cdot \ldots \cdot (\lambda v_n)$
- $(v_1 + v_1') \cdot v_2 \cdot \ldots \cdot v_n = v_1 \cdot v_2 \cdot \ldots \cdot v_n + v_1' \cdot v_2 \cdot \ldots \cdot v_n$
- and similarly linearity in the other factors.

Constructions (exterior power)

$\Lambda^n V$ is the vector space generated by "formal" symbols

$v_1 \wedge v_2 \wedge \dots \wedge v_n$ for all $(v_1, v_2, \dots, v_n) \in V^n$

modulo the relations

- $v_1 \wedge \dots \wedge v_i \wedge \dots \wedge v_j \wedge \dots \wedge v_n = - v_1 \wedge \dots \wedge v_j \wedge \dots \wedge v_i \wedge \dots \wedge v_n$ (for all i,j).
- $\lambda (v_1 \wedge v_2 \wedge \dots \wedge v_n) = (\lambda v_1) \wedge \dots \wedge v_n = \dots = v_1 \wedge \dots \wedge (\lambda v_n)$
- $(v_1 + v_1') \wedge v_2 \dots \wedge v_n = v_1 \wedge v_2 \wedge \dots \wedge v_n + v_1' \wedge v_2 \wedge \dots \wedge v_n$
- and similarly linearity in the other factors.

Remarks about $\Lambda^n V$

Note that $v_1 \wedge v_2 \wedge \dots \wedge v_n = 0$ if $v_i = v_j$ for some $i \neq j$.

For general permutations σ we have

$$v_1 \wedge v_2 \wedge \dots \wedge v_n = \operatorname{sgn}(\sigma) \cdot v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge \dots \wedge v_{\sigma(n)}$$

sign of the permutation.

Universal properties

By construction we get the universal properties that we were looking for:

$$\mathrm{Hom}(\mathrm{Sym}^n V, W) \cong \{\text{sym. multilinear maps } V^n \to W\}$$

and

$$\mathrm{Hom}(\Lambda^n V, W) \cong \{\text{alt. multilinear maps } V^n \to W\}$$

Bases

Let $\{v_1, \ldots, v_k\}$ be a basis of V.

Then

$$\left\{ v_{i_1} \cdot v_{i_2} \cdot \ldots \cdot v_{i_n} \;\middle|\; \begin{array}{l} (i_1, i_2, \ldots, i_n) \in \{1, 2, \ldots, k\}^n \\ \text{such that } i_1 \leq i_2 \leq \ldots \leq i_n \end{array} \right\}$$

(duplicates are allowed)

is a basis of $\mathrm{Sym}^n V$, and

$$\left\{ v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_n} \;\middle|\; \begin{array}{l} (i_1, i_2, \ldots, i_n) \in \{1, 2, \ldots, k\}^n \\ \text{such that } i_1 < i_2 < \ldots < i_n \end{array} \right\}$$

(duplicates are not allowed)

is a basis of $\Lambda^n V$.

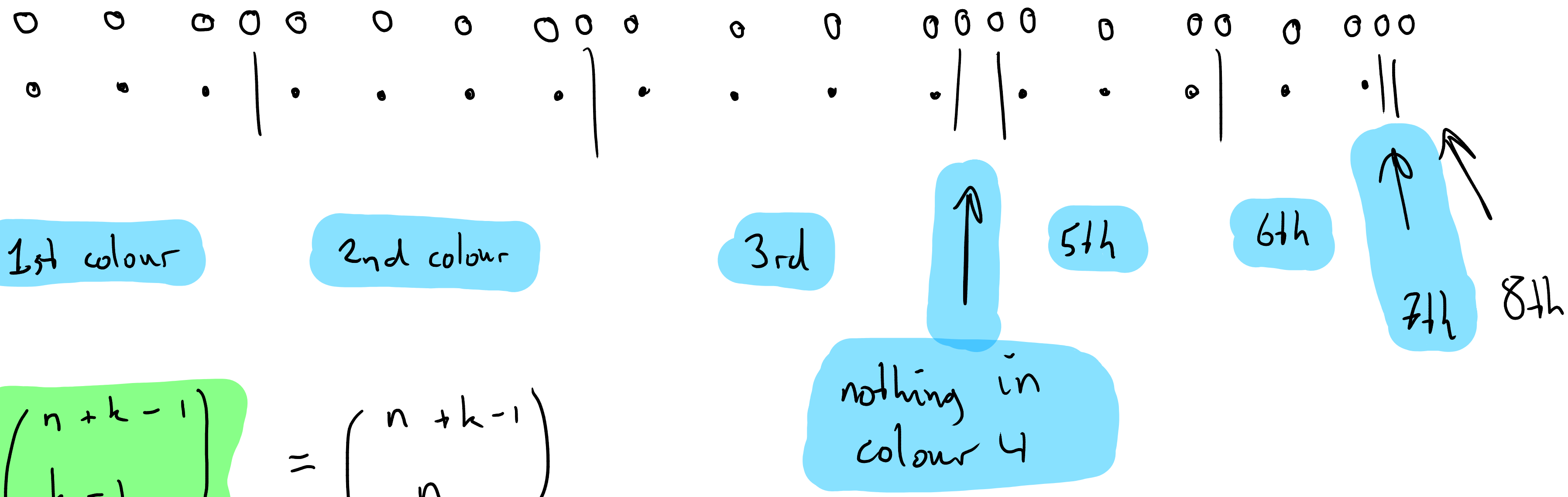
Dimensions

$\dim(V) = k$.

What is $\dim(\mathrm{Sym}^n V)$?

n positions, k basis elements to choose from

n marbles, k colours, how many ways to paint them?



$$\dim = \binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

What is $\dim(\Lambda^n V)$?

We have k basis vectors for V.

We can use each vector only once (since $v_i \wedge v_i = 0$)

We need to choose n of them.

Order doesn't matter.

Hence $\dim = \binom{k}{n}$.

The case $n=2$

$$\dim(\mathrm{Sym}^2 V) = \binom{2+k-1}{k-1} = \binom{k+1}{2} = \frac{k(k+1)}{2}$$

$$\dim(\Lambda^2 V) = \binom{k}{2} = \frac{k(k-1)}{2}$$

Note that $\frac{k(k+1)}{2} + \frac{k(k-1)}{2} = \frac{2k^2}{2} = \dim(V \otimes V)$.

$$\frac{2k^2}{2} = k^2$$

Exercise Show that $V \otimes V \cong \mathrm{Sym}^2 V \oplus \Lambda^2 V$.