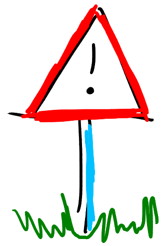
Standard cyclic modules

Let $\mathfrak{g}$ be a semisimple Lie algebra, with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.
Write R for the corresponding root system, and choose a base $\Delta \subset R$.

Let V be a representation. In this lecture we do not assume that V is finite-dimensional.



Definition A maximal vector (of weight λ) for V is a nonzero vector $v \in V_\lambda$ that is killed by $\mathfrak{g}_\alpha$ for all roots $\alpha > 0$ (equivalently, for all $\alpha \in \Delta$).
These are also called highest weight vectors.

Suppose that $v \in V$. In the past (for sl_2 and sl_3) we have looked at subrepresentations generated by succesive applications of X_α or Y_α, where α ranged over Δ or R^+.

We now have more languge to say these things.

<u>Definitions</u> We will write N^+ for the nilpotent subalgebra $\bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$ and N^- for $\bigoplus_{\alpha \in R^-} \mathfrak{g}_\alpha$.

The Borel subalgebra $B(\Delta)$ is the solvable subalgebra $\mathfrak{h} \oplus N^+$.

Definition We say that V is standard cyclic (of weight λ) if there is a maximal vector v^+ of weight λ, such that v^+ generates V. In other words $V = U(\mathfrak{g}) \cdot v^+$, where $U(\mathfrak{g})$ is the universal enveloping algebra.

Warning: If V and W are standard cyclic of weight λ, this does not imply $V \cong W$. However, if V and W are irreducible, then we will see that $V \cong W$ does follow.

For each $\alpha \in R^+$, choose $X_\alpha \in \mathfrak{g}_\alpha$ and $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $(X_\alpha, H_\alpha, Y_\alpha)$ is an $\mathfrak{sl}_2$-triple.

Let $\{\beta_1, \dots, \beta_m\}$ be the set R^+ of positive roots.

<u>Theorem</u> (i) V is spanned by the vectors $Y_{\beta_1}^{i_1} Y_{\beta_2}^{i_2} \cdots Y_{\beta_m}^{i_m} \cdot v^+$ $(i_j \in \mathbb{N})$

(ii) V is the direct sum of its weight spaces.

(iii) The weights of V are of the form $\mu = \lambda - \sum_{\alpha \in \Delta} k_\alpha \alpha$, $k_\alpha \in \mathbb{N}$; that is: $\mu \preceq \lambda$.

(iv) For every $\mu \in \mathfrak{h}^*$, V_μ is finite-dimensional and $\dim V_\lambda = 1$.

(v) Every subrepresentation of V is a direct sum of its weight spaces.

(vi) V is indecomposable, with a unique maximal proper subrepresentation.

(vii) Every homomorphic image of V is either trivial or standard cyclic of weight λ.

<u>Proof</u> (i) Follows from the Poincaré–Birkhoff–Witt theorem, which we did not cover in this course. But intuitively it is clear that we only need to consider Y_α's since the X_α kill v^+ by assumption.

(ii) This follows directly from (i).

(iii) The weight of $Y_{\beta_1}^{i_1} \cdots Y_{\beta_m}^{i_m} v^+$ is $\lambda - \sum i_j \beta_j$.

But every β_j is a sum of simple roots. This proves (iii).

(iv) There are only finitely many $(i_1, \dots, i_m)$ with $i_j \in \mathbb{N}$ such that $\lambda - \sum i_j \beta_j$ equals a given weight μ. In particular $\sum i_j \beta_j = 0$ forces $i_j = 0$ for all j.

Hence $\dim V_\lambda = 1$.

(v) Suppose that W is a subrepresentation of V. Fix $w \in V$.

Write $w = v_1 + \dots + v_n$, with $v_i \in V_{\mu_i}$.

We are done if we show $v_i \in W$. Suppose this is not the case.

Without loss of generality assume that $v_i \notin W$ for all i,

and assume that n is minimal with this property.

(Otherwise, consider a smaller sum.)

Choose $H \in \mathfrak{h}$ such that $\mu_1(H) \neq \mu_2(H)$.

Note that $H(w) \in W$, since W is a subrepresentation.

Also note that $H(w) = \sum H(v_i) = \sum \mu_i(H) v_i$.

Finally $(H - \mu_1(H) \cdot Id)(w) \in W$ and

$$(H - \mu_1(H) \cdot Id)(w) = \sum \mu_i(H) v_i - \sum \mu_1(H) v_i = (\mu_2(H) - \mu_1(H)) v_2 + \dots + (\mu_n(H) - \mu_1(H)) v_n$$

Now we are done. Because we assumed n minimal, we know that

$$(\mu_i(H) - \mu_1(H))v_i \in W.$$

Since $\mu_1(H) \neq \mu_2(H)$, this shows $v_2 \in W$, which contradicts our assumption.

(vi) If W is a proper submodule, then W is a direct sum of weight spaces that don't include V_λ.

So the direct sum of such submodules will also not contain V_λ.

Hence the direct sum of all proper submodules is the unique maximal proper submodule.

In particular, V is indecomposable: if it were a direct sum of proper submodules, those would all be contained in the maximal proper submodule.

(vii) This is immediate from the definition. $\blacksquare$

In particular, the quotient of V by its maximal proper subrepresentation is an irreducible standard cyclic module of weight λ.

<u>Corollary</u> If V is irreducible, then there is a unique maximal vector up to scalar multiples.

<u>Proof</u> By assumption $V = U(\mathfrak{g}) \cdot v^+$. Suppose that w^+ is another maximal vector with weight λ'. Then $U(\mathfrak{g}) \cdot w^+$ is a subrepresentation, and since V is irreducible and $w^+ \neq 0$, this forces $U(\mathfrak{g}) \cdot w^+ = V$. By the theorem, we get $\lambda' \leq \lambda$ and $\lambda \leq \lambda'$. Hence $\lambda' = \lambda$. Since $\dim V_\lambda = 1$, we are done. $\blacksquare$

Theorem Let V and W be standard cyclic modules of weight λ. Assume both V and W are irreducible. Then $V \cong W$.

Proof Consider $X = V \oplus W$. Suppose v^+ and w^+ are maximal vectors of V and W respectively. Then $x^+ = (v^+, w^+)$ is a maximal vector of X of weight λ.

Let Y be the subrepresentation of X generated by x^+, so that Y is standard cyclic.

Let $p : Y \to V$ and $q : Y \to W$ be the projections, which are clearly morphisms of representations. Note that $p(x^+) = v^+$ and $q(x^+) = w^+$.

Hence p and q are surjective, since V and W are irreducible.

But Y is standard cyclic and has a unique irreducible quotient. Hence $V \cong W$. ∎