Irreducible representations of $\mathfrak{sl}_3$

Recall from last time:

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \,\middle|\, a_1 + a_2 + a_3 = 0 \right\}$$

Commuting semisimple endomorphisms admit a basis of common eigenvectors

Preservation of Jordan decomposition:

If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation, $X \in \mathfrak{g}$, and $\mathrm{ad}(X)$ is semisimple then $\rho(X)$ is semisimple

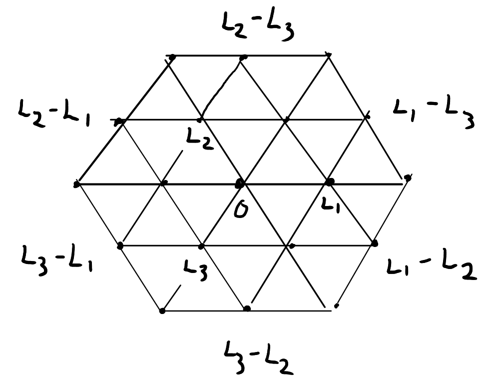
Let V be an irreducible representation of $\mathfrak{sl}_3$.

We find

$$V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha .$$

similar to the decomposition

$$\mathfrak{sl}_3 = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha .$$



How does $\mathfrak{g}_\alpha$ act on V_β?

Suppose $X \in \mathfrak{g}_\alpha$ and $v \in V_\beta$.

Pick some $H \in \mathfrak{h}$. Then we compute

$$
\begin{aligned}
H(X(v)) &= X(H(v)) + [H,X](v) \\
&= X(\beta(H)\cdot v) + \alpha(H)\cdot X(v) \\
&= \beta(H)\cdot X(v) + \alpha(H)\cdot X(v) \\
&= (\alpha+\beta)(H)\cdot X(v)
\end{aligned}
$$

Conclusion: $X(v) \in V_{\alpha+\beta}$.

Some terminology

$$V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$$

$v \in V_\alpha$

weight space

weight

weight vector

$$\mathfrak{sl}_3 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha$$

$X \in \mathfrak{g}_\alpha$

root space

roots

$R = \{ \alpha \in \mathfrak{h}^* \mid \alpha \text{ is a root} \}$

$\Lambda_R = \mathbb{Z} \cdot R \subset \mathfrak{h}^*$ is the root lattice

Let $P(V)$ be the set of weights of V.

P = "poids"

Note: $P(V)$ is finite

Suppose that $\alpha, \beta \in P(V)$. Then there must be a finite sequence of weights $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$ $\alpha_i \in P(V)$ such that $\alpha_{k+1} - \alpha_k$ is a root (so $= L_i - L_j$).

Reason: V is irreducible

In the case of $\mathfrak{sl}_2$ we picked the "largest" weight

$$\{\alpha,\ \alpha+2,\ \alpha+4,\ \dots,\ \alpha+2n\}$$

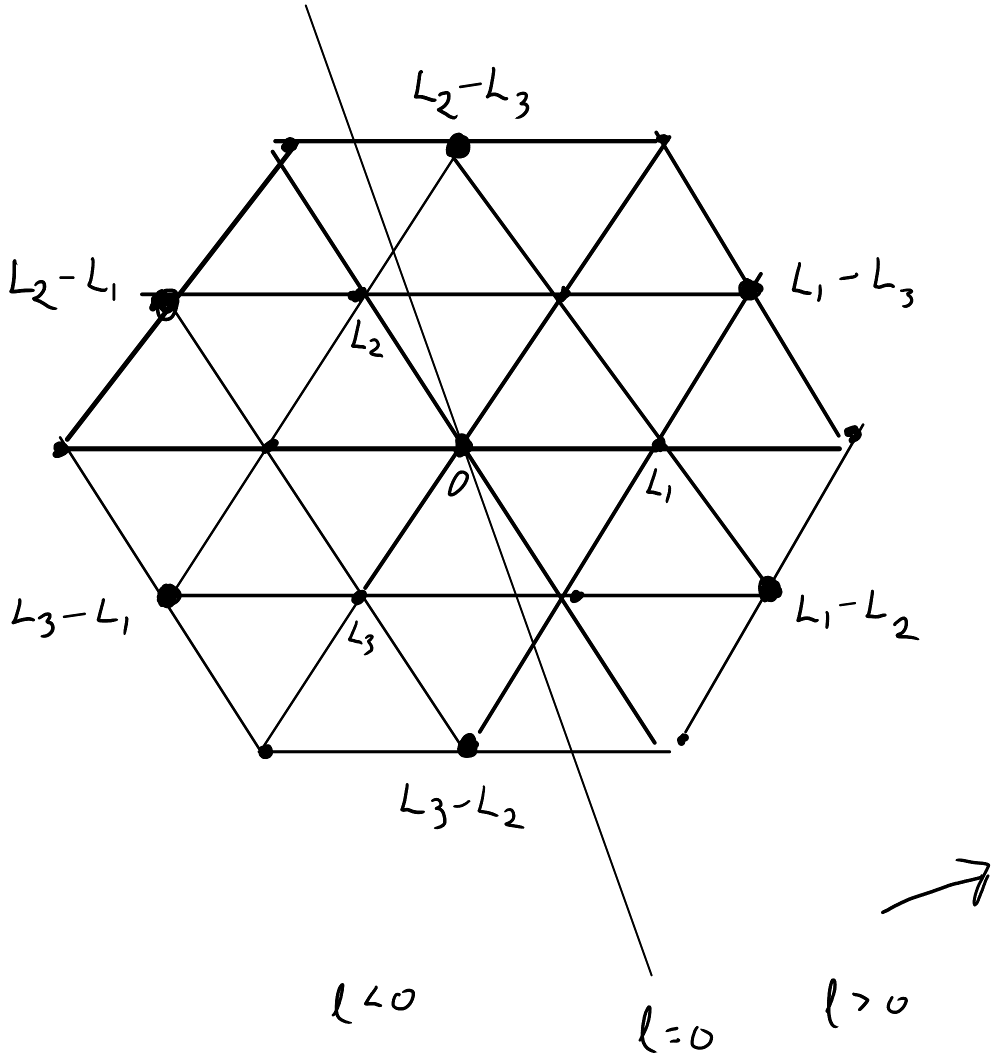
What can we do here?

We will choose a partial ordering of $\mathfrak{h}^*$:

Choose $\ell : \Lambda_R \to \mathbb{R}$ linear and injective and put $\alpha \leq \beta$ if

(i) $\beta - \alpha \in \Lambda_R$, and

(ii) $\ell(\beta - \alpha) \geq 0$



Result: we get a total ordering of Λ_R and hence of $P(V)$

since $\beta - \alpha \in \Lambda_R$ for all $\alpha, \beta \in P(V)$.

We call the maximal element α of $P(V)$ the

heighest weight of V, and nonzero $v \in V_\alpha$ are heighest weight vectors.

Claim: $\dim(V_\alpha) = 1$, in particular every heighest

weight vector generates V as representation.

Proof: next lecture.