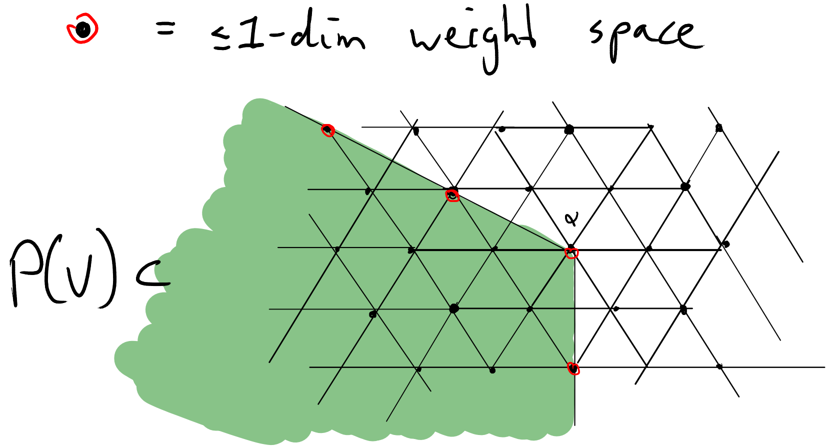
V is generated (as vector space) by the images of v under succesive applications of E_{32}, E_{31}, and E_{21}.



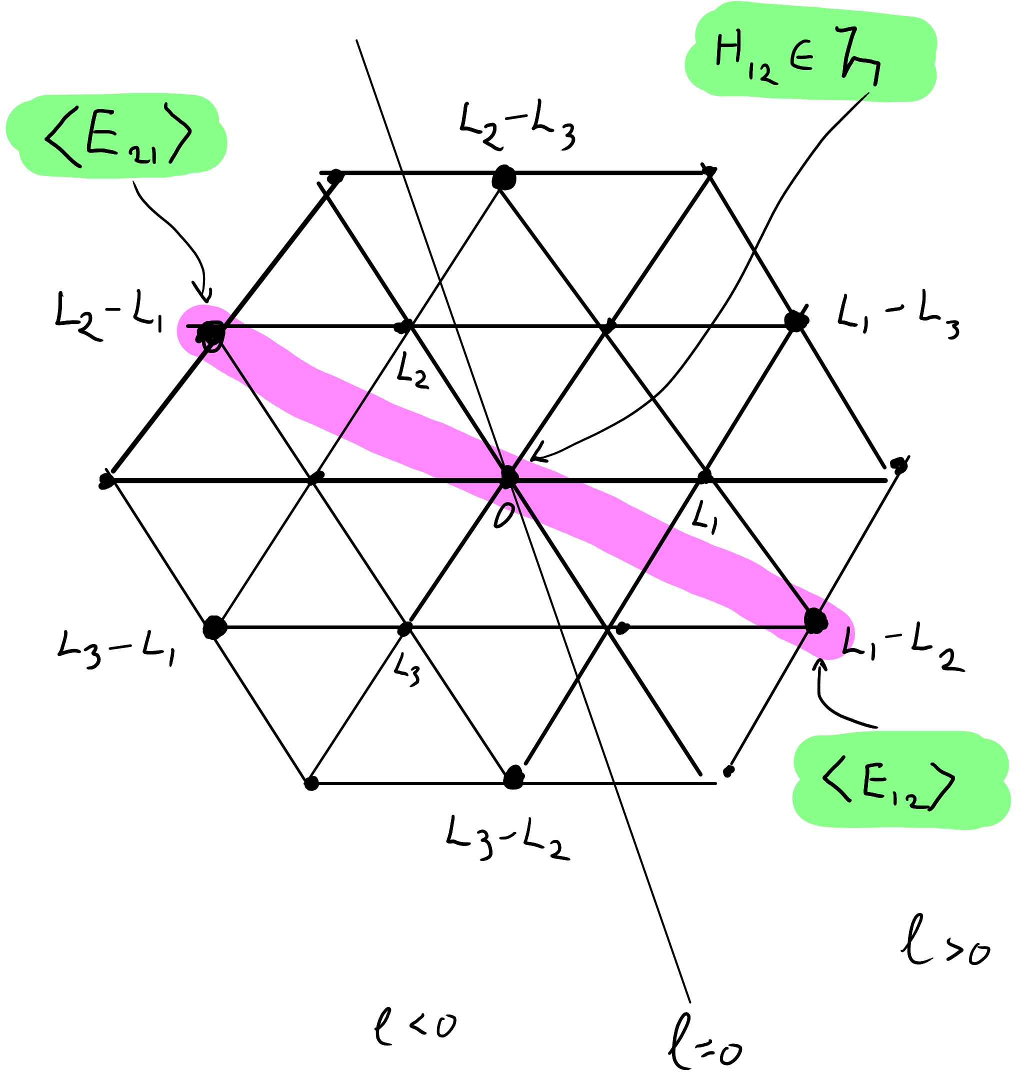
In this lecture:

- ☐ Determine the shape of $P(V)$

Observations

(i) The root space $\mathfrak{g}_{L_i - L_j}$ is generated by E_{ij}.

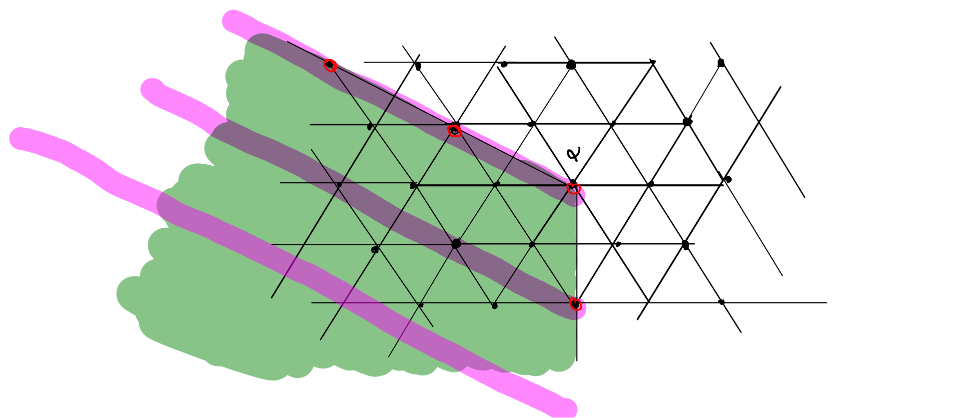
(ii) $H_{ij} = [E_{ij}, E_{ji}] = E_{ii} - E_{jj}$



Fundamental ingredient

$$\begin{array}{ccc} E_{12}, & E_{21}, & H_{12} \\ \wr & \wr & \wr \\ X & Y & H \end{array} \in \mathfrak{sl}_3 \text{ generate}$$

a subalgebra isomorphic to $\mathfrak{sl}_2$.



E_{12}

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

E_{21}

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

H_{12}

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

sl_2

$$\begin{pmatrix} H_{12} & E_{12} & E_{13} \\ E_{21} & H_{23} & E_{23} \\ E_{31} & E_{32} & \end{pmatrix}$$

H_{13}

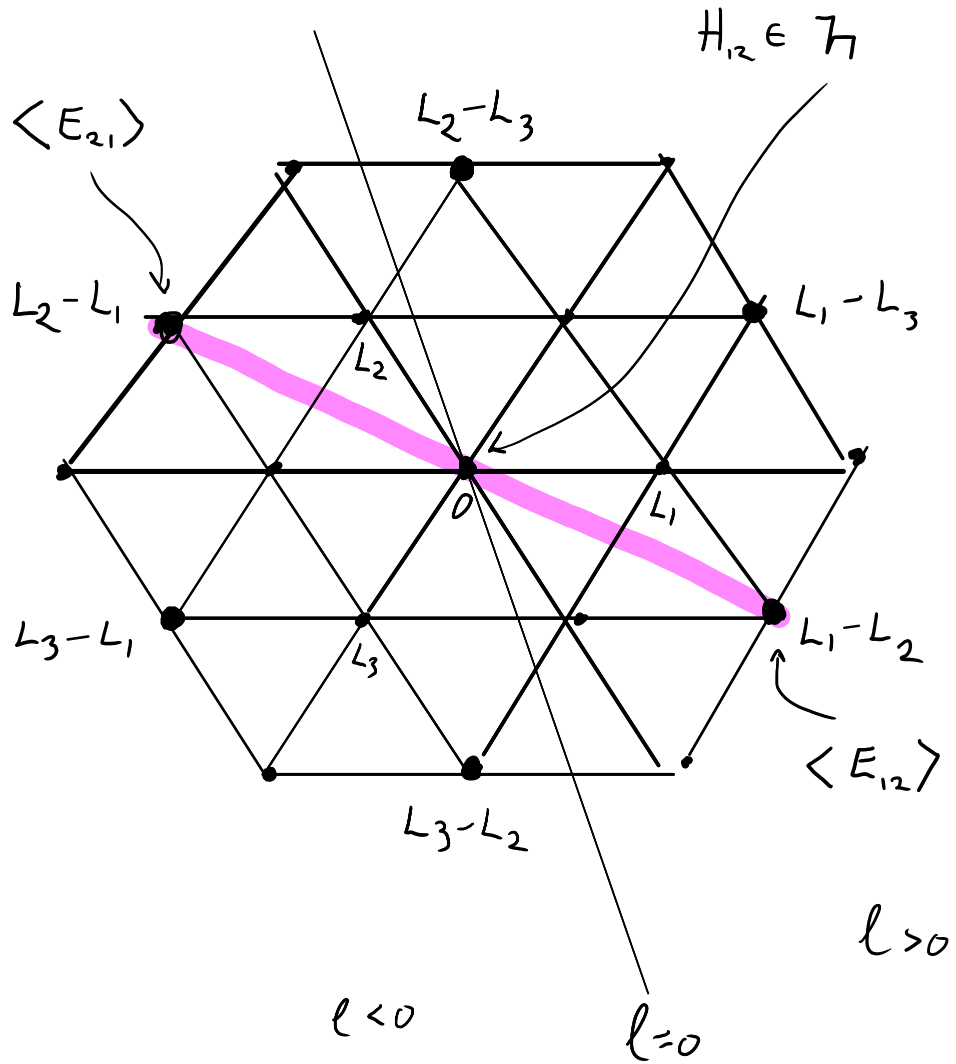
Let $\mathfrak{s}_{L_1-L_2}$ be the subalgebra generated by E_{12}, E_{21}, H_{12}.

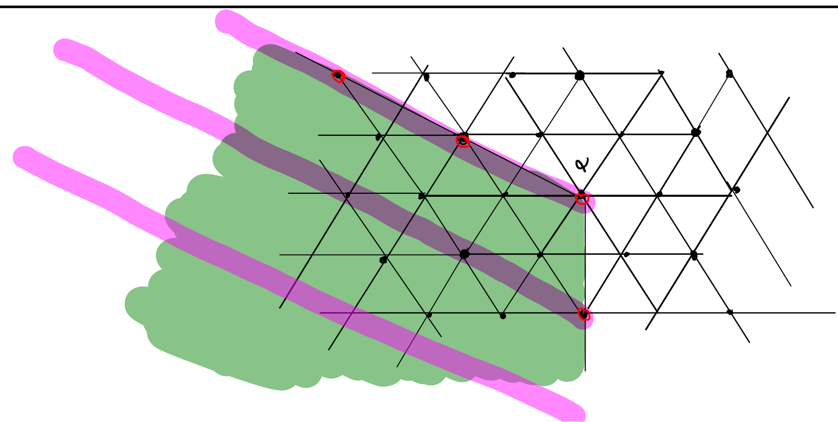
Let $\beta \in P(V)$ be a weight of V.

The subspace

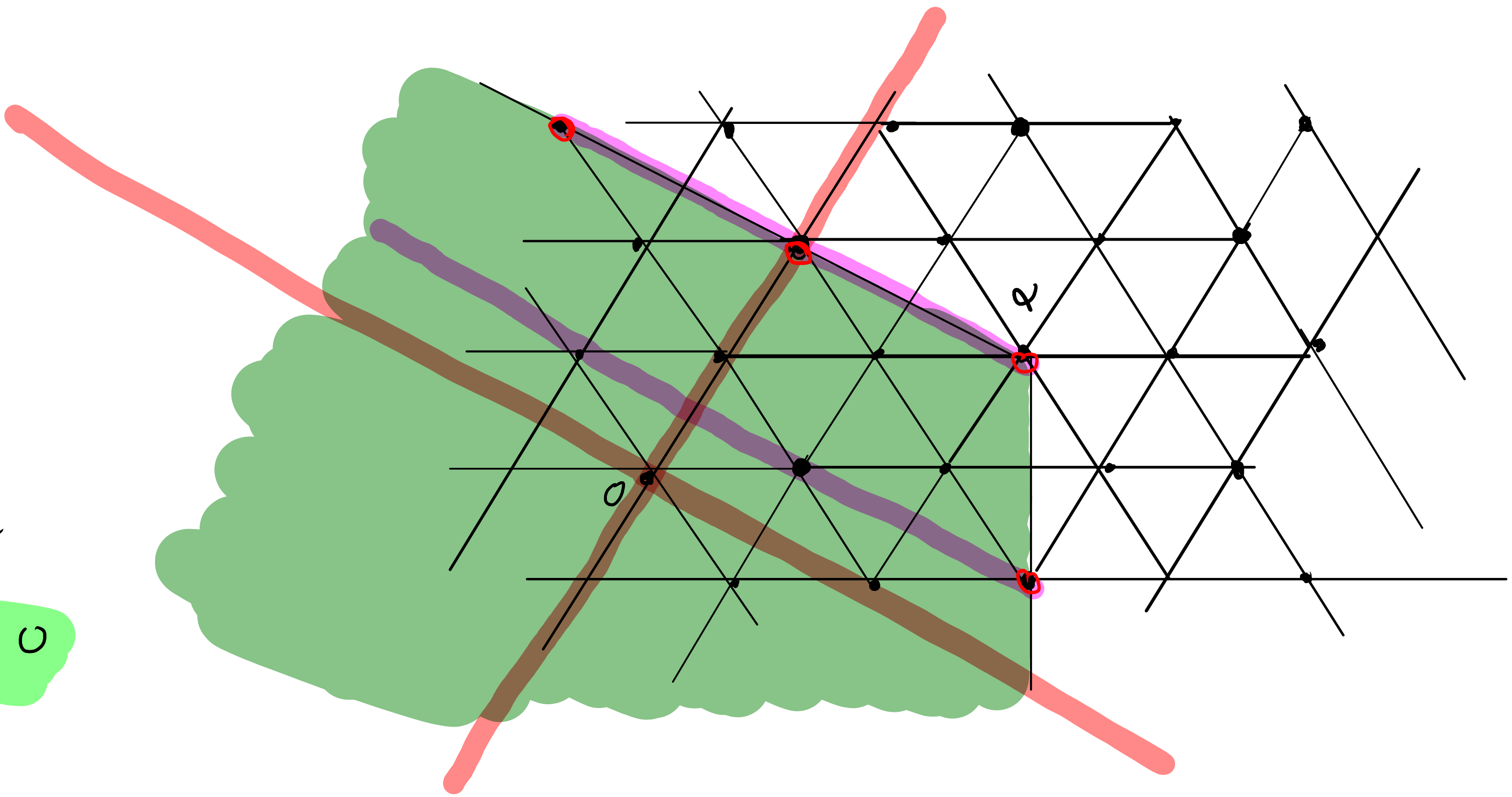
$$W = \bigoplus_{k \in \mathbb{Z}} V_{\beta + k(L_1-L_2)}$$

is stable under $\mathfrak{s}_{L_1-L_2}$.





Conclusion The subspace $W = \bigoplus_k V_{\beta + k(L_1 - L_2)}$ has an eigenspace decomposition for the action of H_{12} that consists of a string of integers symmetric around 0.



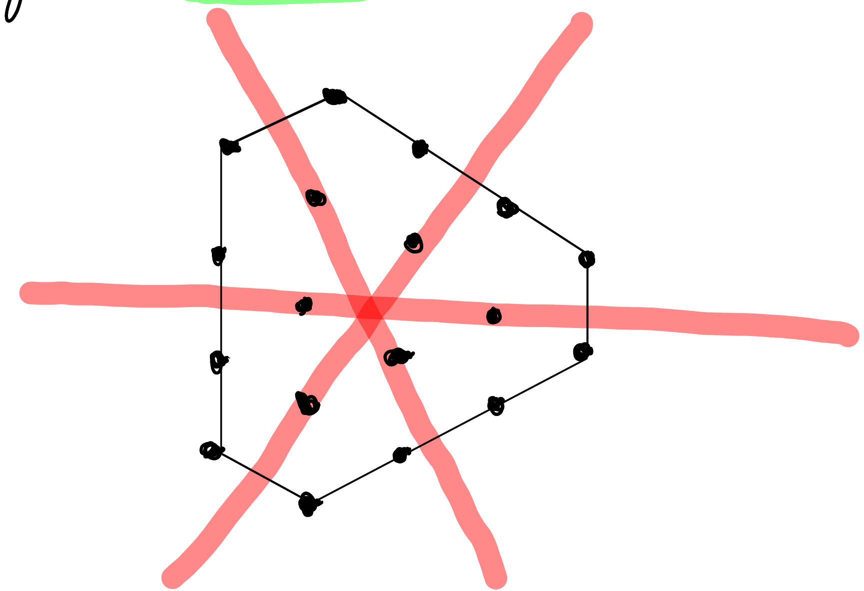
The line $\langle H_{12}, L \rangle = 0$

We can play the exact same game with the other two sl_2-triples: (E_{23}, E_{32}, H_{23}) (E_{13}, E_{31}, H_{13})

Hence we get additional symmetries around

$\langle H_{23}, L\rangle = 0$ and $\langle H_{13}, L\rangle = 0$

Therefore $P(V)$ must look like a hexagon



All weight spaces on the boundary are 1-dimensional.

What about the other weight spaces?