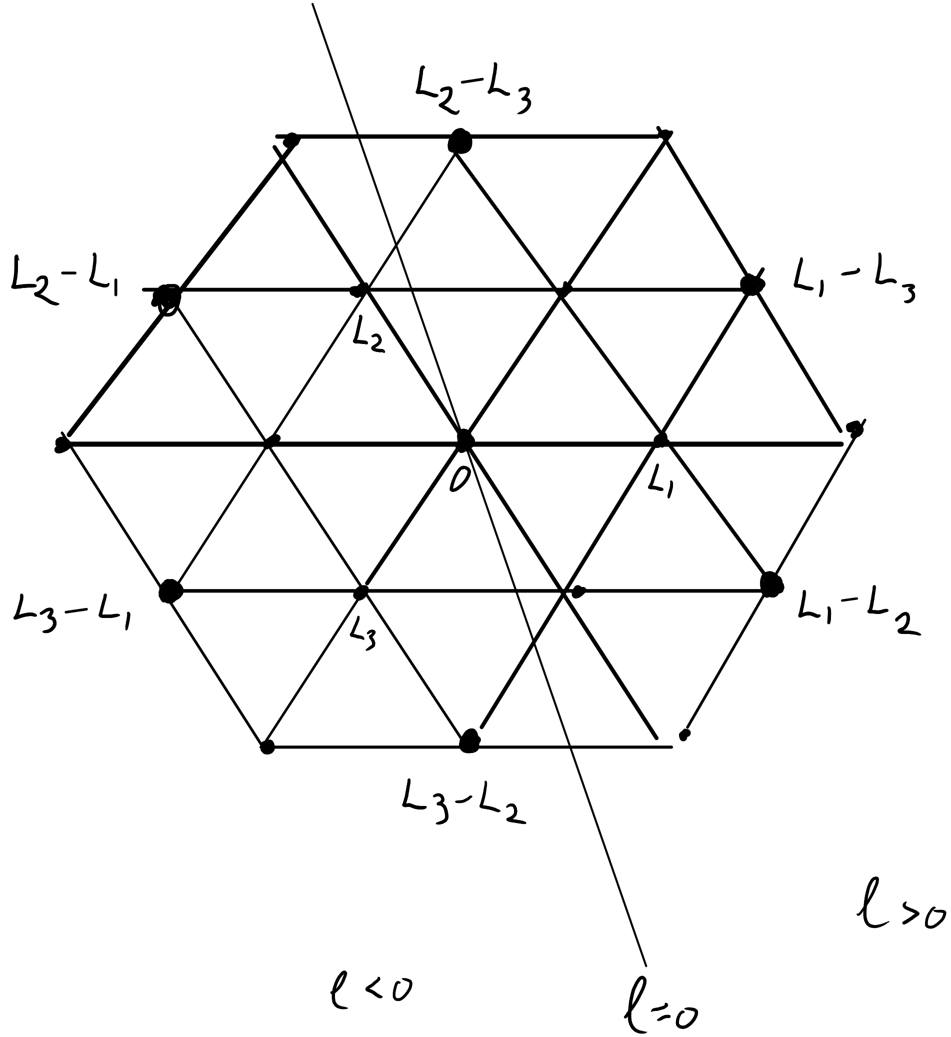
Observations

(i) The root spaces $\mathfrak{g}_{L_i - L_j}$

is generated by E_{ij}.

(ii) Positive roots: $L_i - L_j$ for $i<j$

$$\{L_1 - L_2,\ L_1 - L_3,\ L_2 - L_3\}$$



Notation: we will write H_{ij} for

$$[E_{ij}, E_{ji}] = E_{ii} - E_{jj}.$$

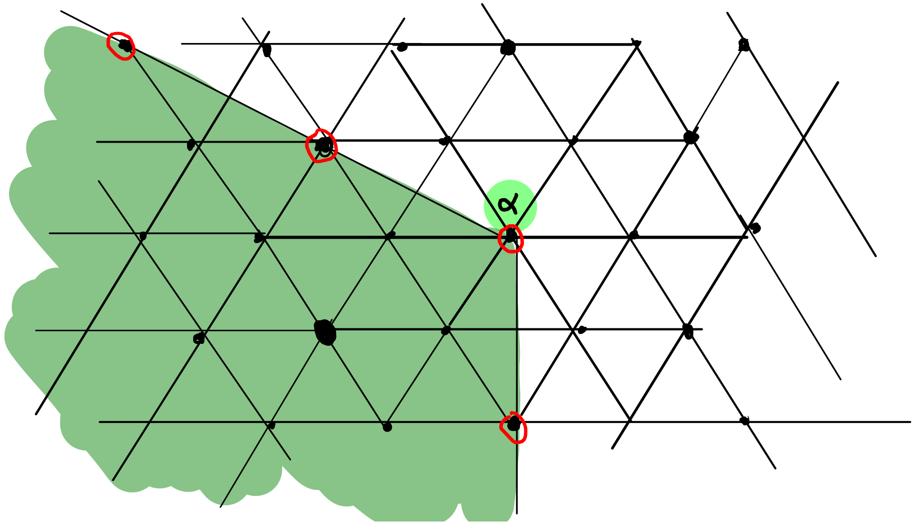
Note: $H_{ij} \in \mathfrak{h}$

Claim Let V be an irreducible fin. dim. representation of $\mathfrak{sl}_3$, and let $\alpha \in P(V)$ be the highest weight, and $v \in V_\alpha$ a highest weight vector.

Then V is generated (as vector space) by the images of v under succesive applications of E_{32}, E_{31}, and E_{21}.

Corollary

$P(V) \in$



Weight spaces on the edge ⊙ must be ≤ 1-dimensional.

<u>Proof</u> Let W be the subspace of V generated by images of v under applications of E_{32}, E_{31}, and E_{21}.

We want to show that W is a subrepresentation.

We need to check that $E_{ij}(W) \subset W$ for all $i \neq j$.

If $i > j$, then this is true by construction of W.

Now assume $i < j$. Note that $E_{13} = [E_{12}, E_{23}]$

hence we can assume $(i,j) = (1,2)$ or $(i,j) = (2,3)$.

Since v is a highest weight vector we find $E_{ij}(v) = 0$.

W is generated by the images of v under succesive applications of E_{21} and E_{32}, because $E_{31} = [E_{32}, E_{21}]$.

We continue the proof by induction on the number of those applications.

Let w be a word of length $\leq n$ in the letters E_{21} and E_{32} and let W_n be the subspace of W generated by such words.

Note: $\bigcup_{n \geq 0} W_n = W$. We will show $E_{12}(W_{n+1}) \subset W_n$

$E_{23}(W_{n+1}) \subset W_n$

Note that $w(v) \in V_\beta$ for some weight $\beta \in P(V)$

We compute $E_{12}(E_{21}(w(v))) = \underbrace{E_{21}(E_{12}(w(v)))}_{\in E_{21}(W_{n-1})} + \underbrace{[E_{12}, E_{21}]}_{\in \mathfrak{h}}(w(v))$

$$\in W_n + \beta(H_{12}) \cdot w(v)$$

$$\in W_n$$

Similarly $E_{12}(E_{32}(w(v))) = \underbrace{E_{32}(E_{12}(w(v)))}_{\in E_{32}(W_{n-1})} + \underbrace{[E_{12}, E_{32}]}_{=0}(w(v))$

$$\in W_n$$

This shows $E_{12}(W_{n+1}) \subset W_n$. In the same way $E_{23}(W_{n+1}) \subset W_n$.

This shows $E_{12}(W_{n+1}) \subset W_n$. In the same way $E_{23}(W_{n+1}) \subset W_n$.

Conclusion: $E_{ij}(W_n) \subset \begin{cases} W_{n-1} & \text{if } i<j \\ W_{n+1} & \text{if } i>j \end{cases}$

Hence $E_{ij}(W) \subset W$, and we conclude that W is an sl_3-subrepresentation of V. Hence $W = V$.

V is generated (as vector space) by the images of v under succesive applications of E_{32}, E_{31}, and E_{21}.

$\odot$ = ≤ 1-dim weight space

$P(V) \subset$