The picture for sl_3

After sl_2, the easiest simple Lie algebra is sl_3.

We want to study it in more detail, so that we can use it as a guide when we look at the general case.

Recall from our study of sl_2:

- we have a basis $\{H, X, Y\}$
- $[H, X] = 2X$ $\quad [H, Y] = -2Y$ $\quad [X, Y] = H$

What should play the role of H in sl_3 ?

We will consider $\mathfrak{h}$, the abelian subalgebra of diagonal matrices:

$$\left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \;\middle|\; a_1 + a_2 + a_3 = 0 \right\}$$

Note that $\dim(\mathfrak{h}) = 2$.

We cannot talk about eigenvectors of $\mathfrak{h}$ in the way that X and Y were eigenvectors of $ad(H)$. Instead, we will consider $\alpha : \mathfrak{h} \to K$ and elements X satisfying $[H, X] = \alpha(H) \cdot X$ for all $H \in \mathfrak{h}$.

Recall from the exercises

commuting diagonalisable operators

are simultaneously diagonalisable

Put differently:

given a set of commuting semisimple endomorphisms

there exists a basis consisting of eigenvectors for

all these endomorphisms

Let's apply that to $\mathrm{ad}: \mathfrak{h} \longrightarrow \mathfrak{gl}(\mathfrak{sl}_3)$

It means that there is a decomposition

$$\mathfrak{sl}_3 = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$$

such that for all $X \in \mathfrak{g}_\alpha$ and $H \in \mathfrak{h}$ we have $[H, X] = \alpha(H) \cdot X$

Note that $\mathfrak{g}_0 = \mathfrak{h}$.

What do the other $\mathfrak{g}_\alpha$ look like?

There are some natural elements of $\mathfrak{h}^*$

Recall $\quad \mathfrak{h} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \;\middle|\; a_1 + a_2 + a_3 = 0 \right\}$

Denote by $L_i : \mathfrak{h} \to K$ the map $\quad H = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \longmapsto a_i$

Now compute the coefficients of $[H, M]$ for $H \in \mathfrak{h}$.

HM multiplies the i-th row of M by a_i

MH multiplies the i-th column of M by a_i

Coefficients of $[H, m] = HM - MH$ are $a_i m_{ij} - a_j m_{ij} = (a_i - a_j) m_{ij}$

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Conclusion: the matrix E_{ij} coefficient (i,j) is 1 otherwise 0

is an eigenvector of $\mathrm{ad}(H)$ for all $H \in \mathfrak{h}$

with eigenvalue $L_i(H) - L_j(H)$.

In other words, for $i,j \in \{1,2,3\}$ and $i \neq j$ (so that $\mathrm{tr}(E_{ij}) = 0$)

put $\alpha := L_i - L_j$. Then $E_{ij} \in \mathfrak{g}_\alpha$.

For dimension reasons $\mathfrak{g}_\alpha = \langle E_{ij} \rangle$.

$\mathfrak{h}^*$

Picture of $\mathfrak{sl}_3$