Simple roots and Weyl chambers

Let $R \subset E$ be a root system of rank ℓ, with Weyl group $\mathcal{W}$.

A **base** of R is a subset $\Delta \subset R$ such that

- Δ is a basis for E
- Every root $\beta \in R$ can be written as

$$\beta = \sum_{\alpha \in \Delta} n_\alpha \alpha, \qquad n_\alpha \in \mathbb{Z}$$

where either all the n_α are nonnegative or all nonpositive.

The roots in Δ are called simple roots.

Since Δ is a basis of E, the expression $\beta = \sum_{\alpha \in \Delta} n_\alpha \alpha$ is unique. We define the height of β via

$$\mathrm{ht}(\beta) = \sum_{\alpha \in \Delta} n_\alpha$$

If $\mathrm{ht}(\beta) > 0$, then we call β a positive root, and if $\mathrm{ht}(\beta) < 0$, then β is negative.

Notation: $R^+ = \{\beta \mid \mathrm{ht}(\beta) > 0\}$ and $R^- = \{\beta \mid \mathrm{ht}(\beta) < 0\}$

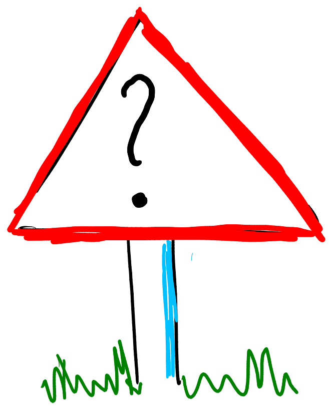
Note: $-R^+ = R^-$. If $\alpha, \beta \in R^+$, then $\alpha + \beta \in R^+$.

As a result we get a partial ordering of E, similar to what we saw in the example of $\mathfrak{sl}_3$: for $x, y \in E$, we define

$$x < y \quad \text{iff} \quad y - x \text{ is a } \mathbb{N}\text{-linear combination of simple roots}$$

As we saw with $\mathfrak{sl}_3$, this ordering was a powerful tool in the analysis of the Lie algebra and its representations.

But first we need to answer a crucial question!

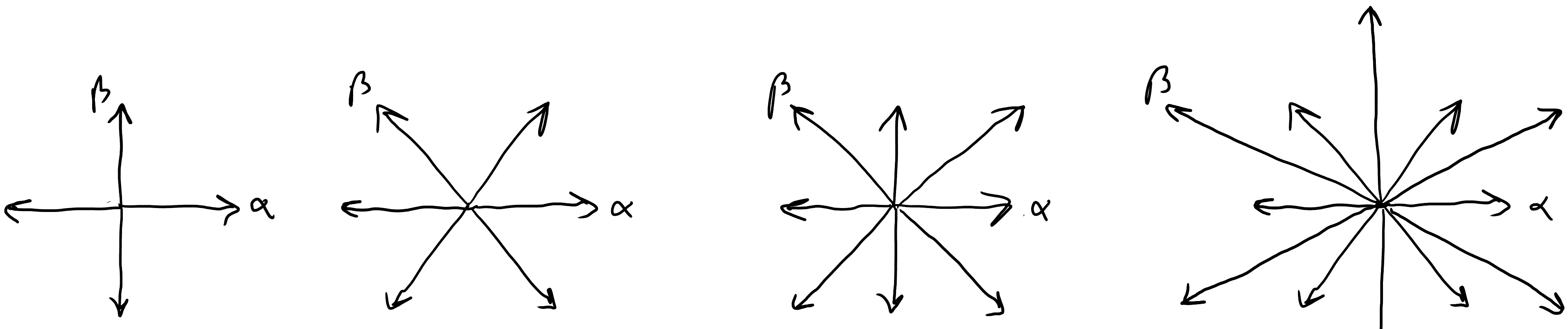


Q: Does a base Δ always exist?

To answer this existence question we start with the following observation:

Lemma If $\alpha \neq \beta$ are two simple roots, then the angle between α and β is obtuse. In other words $(\alpha, \beta) \leq 0$.

Proof Otherwise $(\alpha,\beta) > 0$, and hence $\alpha - \beta$ is a root by one of the results from last time. But $\alpha - \beta$ cannot be a root by the condition on Δ. $\blacksquare$



$\Delta = \{\alpha, \beta\}$ is a base for these root systems of rank 2.

For $x \in E$, let $R^+(x)$ be $\{\alpha \in R \mid (x, \alpha) > 0\}$.

So $R^+(x)$ consists of the roots lying on the positive side of the hyperplane P_x.

Now call x singular if $x \in P_\alpha$ for some $\alpha \in R$, and otherwise call x regular. Clearly, regular elements exist.

If x is regular, then $R = R^+(x) \cup -R^+(x)$.

Finally $\alpha \in R^+(x)$ is decomposable if $\alpha = \beta_1 + \beta_2$, for some $\beta_i \in R^+(x)$, and otherwise α is indecomposable.

Theorem The set $\Delta(\gamma)$ of indecomposable elements form a base. Every base is obtained in this way.

Proof Step 1: Every root in $R^+(x)$ is an $\mathbb{N}$-linear combination of $\Delta(x)$.

Otherwise, let α be a root not of this form, with the property that (x, α) is as small as possible. Such an α must be decomposable: $\alpha = \beta_1 + \beta_2$.

Hence $(x, \alpha) = (x, \beta_1) + (x, \beta_2)$. But (x, β_1) and (x, β_2) are both positive, hence less than (x, α). Therefore β_1 and β_2 are $\mathbb{N}$-linear combinations of $\Delta(x)$, and so is α.

Step 2: If $\alpha \neq \beta$ are in $\Delta(x)$, then $(\alpha, \beta) \leq 0$. Otherwise $\alpha - \beta$ and $\beta - \alpha$ are roots, one of which is in $R^+(x)$, say $\alpha - \beta$. But then $\alpha = (\alpha - \beta) + \beta$, which contradicts $\alpha \in \Delta(x)$.

Step 3: $\Delta(x)$ is linearly independent, hence a basis.

Suppose that $\sum_{\alpha \in \Delta(x)} r_\alpha \alpha = 0$ $(r_\alpha \in \mathbb{R})$. Now split $\Delta(x)$ into pieces

$$\Delta(x)^+ = \{\alpha \mid r_\alpha > 0\} \qquad \Delta(x)^- = \{\alpha \mid r_\alpha < 0\}$$

Write s_α for $-r_\alpha$.

Then we get $y := \sum_{\alpha \in \Delta(x)^+} r_\alpha \alpha = \sum_{\alpha \in \Delta(x)^-} s_\alpha \alpha$.

Then $(y,y) = \sum_{\substack{\alpha\in\Delta(x)^+\\ \beta\in\Delta(x)^-}} r_\alpha s_\beta (\alpha,\beta) \leq 0$

since $r_\alpha, s_\beta > 0$ and $(\alpha,\beta)\leq 0$ by step 2.

Consequently $y=0$.

But then $0=(x,y) = \sum_{\alpha\in\Delta(x)^+} r_\alpha (x,\alpha) = \sum_{\beta\in\Delta(x)^-} s_\beta (x,\beta)$.

Since (x,α) and (x,β) are positive by assumption, this forces $r_\alpha = 0 = s_\beta$.

This proves linear independence.

Step 4: We conclude from step 3 and regularity of x, that $\Delta(x)$ is a base.

Step 5: Every base Δ of R is of the form $\Delta(x)$ for some regular $x \in E$.

Choose $x \in E$ such that $(x, \alpha) > 0$ for all $\alpha \in \Delta$.

(Exercise: check that this is possible!)

Then x is regular and the positive (resp. negative) roots of R with respect to Δ must be contained in $R^+(x)$ (resp. $-R^+(x)$).

This forces $R^+(\Delta) = R^+(x)$ and $R^-(\Delta) = -R^+(x)$.

Then Δ must consist of indecomposable elements, so $\Delta \subset \Delta(x)$.

We get $\Delta = \Delta(x)$ since both sets are a basis of E. $\square$

Weyl chambers

In the last step of the proof above, we picked x from the set.

$$\{ x \mid (x,\alpha) > 0 \text{ for all } \alpha \in \Delta \}$$

This is the fundamental Weyl chamber relative to Δ.

Definition The connected components of $E - \bigcup_{\alpha \in R} P_\alpha$ are the Weyl chambers of the root system R.

We have just seen that there is a natural bijection between bases of R and Weyl chambers of R.

With respect to a chosen base Δ, the Weyl chambers are all of the form

$$\{ x \mid \varepsilon_\alpha (x, \alpha) > 0 \text{ for all } \alpha \in \Delta \}$$

where $(\varepsilon_\alpha)_{\alpha \in \Delta}$ is an assignment of ± 1 to each simple root $\alpha \in \Delta$.

Exercise: Let $\sigma \in \mathcal{W}$ be an element of the Weyl group.

Show that $\sigma(\Delta)$ is again a base of R.

Show that the action of the Weyl group on the set of Weyl chambers is compatible with its action on the collection of bases.

<u>Properties of simple roots</u> Fix a base Δ of R.

<u>Lemma</u> If $\alpha \in R$ is positive but not simple, then $\alpha - \beta$ is a positive root for some $\beta \in \Delta$.

<u>Proof</u> Replaying step 3 from the proof above, we find $(\alpha, \beta) > 0$ for some $\beta \in \Delta$ and hence $\alpha - \beta \in R$.
Since α is a $\mathbb{N}$-linear combination of Δ, $\alpha - \beta$ must also be a $\mathbb{Z}$-linear combination with nonnegative coefficients. Hence $\alpha - \beta$ is positive $\blacksquare$

<u>Corollary</u> Every positive root can be written as sum $\alpha_1 + \dots + \alpha_k$ $(\alpha_i \in \Delta)$ such that every partial sum $\alpha_1 + \dots + \alpha_i$ is a positive root.

Lemma Let α be a simple root. Then σ_α permutes $R^+ - \{\alpha\}$.

Proof Let $\beta \neq \alpha$ be a positive root. Write $\beta = \sum_{\gamma \in \Delta} n_\gamma \gamma$.

Since $\beta \neq \alpha$, there must be a $\gamma \neq \alpha$ such that $n_\gamma > 0$.

Now $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$, so only the coefficient of α changes.

This means that $\sigma_\alpha(\beta)$ is positive, and clearly $\sigma_\alpha(\beta) \neq \alpha$. $\blacksquare$

Corollary Put $\delta = \frac{1}{2} \sum_{\beta > 0} \beta$. Then $\sigma_\alpha(\delta) = \delta - \alpha$.

Lemma Let $\alpha_1, \ldots, \alpha_t \in \Delta$ be some roots (not necessarily distinct). Write σ_i for σ_{α_i}. If $\sigma_1 \cdots \sigma_{t-1}(\alpha_t)$ is negative, then for some index $1 \leq s < t$, we have $\sigma_1 \cdots \sigma_{s-1} \sigma_{s+1} \cdots \sigma_{t-1} = \sigma_1 \cdots \sigma_t$.

Proof Write $\beta_i = \sigma_{i+1} \cdots \sigma_{t-1}(\alpha_t)$, for $0 \leq i < t-1$, $\beta_{t-1} = \alpha_t$. Since $\beta_0 < 0$ and $\beta_{t-1} > 0$, we can find a smallest s such that $\beta_s > 0$. Then $\sigma_s(\beta_s) = \beta_{s-1} < 0$. By the previous lemma, this implies $\beta_s = \alpha_s$. Put $\sigma = \sigma_{s+1} \cdots \sigma_{t-1}$, so that $\alpha_s = \beta_s = \sigma(\alpha_t)$. Now we use the rule $\sigma_{\sigma(\alpha_t)} = \sigma \sigma_{\alpha_t} \sigma^{-1}$ (proof below) to conclude $\sigma_s = \sigma \sigma_t \sigma^{-1}$. And hence $\sigma_s \sigma = \sigma \sigma_t$. Multiplying both sides with $\sigma_1 \cdots \sigma_s$ on the left finishes the proof. ∎

On the next page we prove the claim that was made in the proof.

<u>Corollary</u> If $\sigma = \sigma_1 \cdots \sigma_t$ is an expression in $\mathcal{W}$, with $\sigma_i = \sigma_{\alpha_i}$ and t as small as possible, then $\sigma(\alpha_t) < 0$.

Next time:

- more details on the action of $\mathcal{W}$ on the set of Weyl chambers
- $\mathcal{W}$ is an example of a Coxeter group, but as announced before we will not go into the general theory of Coxeter groups.

<u>Claim</u> If $\sigma \in GL(E)$ leaves R invariant, then $\sigma \sigma_\alpha \sigma^{-1} = \sigma_{\sigma(\alpha)}$ for all $\alpha \in R$.

<u>Proof</u> All roots are of the form $\sigma(\beta)$ for some $\beta \in R$, by assumption.

For all $\beta \in R$, we find $\sigma \sigma_\alpha \sigma^{-1}(\sigma(\beta)) = \sigma \sigma_\alpha(\beta) = \sigma(\beta) - \langle \beta, \alpha \rangle \sigma(\alpha)$,

and therefore $\sigma \sigma_\alpha \sigma^{-1}$
- ☐ fixes the hyperplane $P_{\sigma(\alpha)}$
- ☐ sends $\sigma(\alpha)$ to $-\sigma(\alpha)$
- ☐ leaves R invariant.

By exercise 9.5 we conclude that $\sigma \sigma_\alpha \sigma^{-1} = \sigma_{\sigma(\alpha)}$. ∎