Centre

The centre of a Lie algebra $\mathfrak{g}$ is the ideal $Z(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0, \text{ for all } Y \in \mathfrak{g}\}$.

It is the kernel of $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$.

A Lie algebra is called abelian if it is equal to its own centre, so if:

$$[X, Y] = 0 \quad \text{for all } X, Y \in \mathfrak{g}$$

Notation

Suppose that $\mathfrak{a}$ and $\mathfrak{b}$ are two subspaces of $\mathfrak{g}$.

Then we denote by

$$[\mathfrak{a}, \mathfrak{b}]$$

the subspace generated by

$$\{ [x, y] \mid x \in \mathfrak{a}, y \in \mathfrak{b} \}.$$

Exercise: if $\mathfrak{a}$ and $\mathfrak{b}$ are ideal, then so is $[\mathfrak{a}, \mathfrak{b}]$.

Lower central series

$$\mathcal{D}_0\mathfrak{g} = \mathfrak{g}$$

$$\mathcal{D}_{k+1}\mathfrak{g} = [\mathfrak{g}, \mathcal{D}_k\mathfrak{g}]$$

Derived series

$$\mathcal{D}^0\mathfrak{g} = \mathfrak{g}$$

$$\mathcal{D}^{k+1}\mathfrak{g} = [\mathcal{D}^k\mathfrak{g}, \mathcal{D}^k\mathfrak{g}]$$

Upper central series

$$\mathcal{C}_0\mathfrak{g} = 0$$

$$\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathcal{C}_k\mathfrak{g}$$

$$\mathcal{C}_{k+1} = \pi^{-1}(Z(\mathfrak{g}/\mathcal{C}_k\mathfrak{g}))$$

centre

Commutator ideal

$$\mathcal{D}_1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = \mathcal{D}^1\mathfrak{g}$$

$$\parallel$$

$$\mathcal{D}\mathfrak{g}$$

Observations

Note that $C_1\mathfrak{g} = Z(\mathfrak{g})$.

Note that $\mathfrak{g}/D\mathfrak{g}$ is abelian.

__Exercise__: If $f : \mathfrak{g} \to \mathfrak{h}$ is a morphism of Lie algebras, then $f(\mathcal{D}_k \mathfrak{g}) \subset \mathcal{D}_k \mathfrak{h}$ and $f(\mathcal{D}^k \mathfrak{g}) \subset \mathcal{D}^k \mathfrak{h}$ with equalities if f is surjective.

Also show that $\mathcal{C}_k \mathfrak{g} \subset f^{-1}(\mathcal{C}_k \mathfrak{h})$.