Simple and semisimple Lie algebras

A Lie algebra is simple if it has dimension > 1 and it does not have any non-trivial proper ideals.

A Lie algebra is semisimple if it does not have any non-trivial solvable ideals

Recall that the sum of two solvable ideals is solvable.

Hence there is a maximal solvable ideal in every finite-dimensional Lie algebra $\mathfrak{g}$.

It is called the radical, and denoted $\mathrm{Rad}(\mathfrak{g})$.

Exercise The following are equivalent:

$\mathfrak{g}$ is semisimple

$\mathrm{Rad}(\mathfrak{g}) = 0$

$\mathfrak{g}$ has no non-trivial abelian ideals

We get a short exact sequence

$$0 \to \mathrm{Rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/\mathrm{Rad}(\mathfrak{g}) \to 0$$

<u>Exercise</u> Show that $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ is semisimple.

<u>Exercise</u> If $f : \mathfrak{g} \to \mathfrak{h}$ is a morphism of Lie algebras, then $f(\mathrm{Rad}(\mathfrak{g})) \subset \mathrm{Rad}(\mathfrak{h})$.

<u>Exercise</u> Let $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$ be Lie algebras.

Then $\mathrm{Rad}(\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k) = \mathrm{Rad}(\mathfrak{g}_1) \oplus \cdots \oplus \mathrm{Rad}(\mathfrak{g}_k)$.

The short exact sequence

$$0 \to \mathrm{Rad}(\mathfrak{g}) \to \mathfrak{g} \longrightarrow \mathfrak{g}/\mathrm{Rad}(\mathfrak{g}) \to 0$$

completes the "rough" classification.

The classification of semisimple Lie algebras will focus on a better understanding of the quotient $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$.

A representation of $\mathfrak{g}$ is called

indecomposable if it is nontrivial and can not be written as direct sum $V \oplus W$ of subrepresentations

simple if it is nontrivial and does not have nontrivial proper subrepresentations

semisimple if it is a direct sum of simple rep'ns.

Note: simple $\Longleftrightarrow$ indecomposable + semisimple

In the future:

- ☐ every representation of a semisimple Lie algebra is semisimple (as representation)

Corollary:

- ☐ every semisimple Lie algebra is a product of simple Lie algebras