Classification of root systems

Let $R \subset E$ be a root system, with base $\Delta = (\alpha_1, \dots, \alpha_\ell)$.

We have already seen the Cartan matrix before. It is

$$\left(\langle \alpha_i, \alpha_j \rangle \right)_{i,j = 1..\ell}$$

Rank 2 examples:

$A_1 \times A_1 \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad A_2 \quad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad B_2 \quad \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \qquad G_2 \quad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

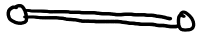
The Cartan matrix determines the root system up to isomorphism.

<u>Proposition</u> Let $R' \subset E'$ be another root system, with base $(\alpha'_1, \dots, \alpha'_\ell)$.
If $\langle \alpha_i, \alpha_j \rangle = \langle \alpha'_i, \alpha'_j \rangle$ for all i,j then the function $\alpha_i \mapsto \alpha'_i$ extends to an isomorphism of root systems.

<u>Proof</u> Exercise. Use the results from last time.

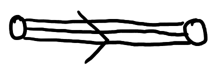
<u>Exercise</u> (Interesting, but nontrivial)
Devise an algorithm that takes as input a Cartan matrix, and gives as output a list of all roots, written as $\mathbb{Z}$-linear combination of the simple roots.

Definition The Coxeter graph of R has Δ as vertex set and vertex α_i is joined to vertex α_j by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges.

Examples $A_1 \times A_1$  B_2 

A_2  G_2 

If all roots have equal length, then the Coxeter graph determines the Coxeter matrix, since then $\langle \alpha_i, \alpha_j \rangle = \langle \alpha_j, \alpha_i \rangle$.

In case of different lengths, we can add extra decoration, to form the Dynkin diagram: for every double or triple edge we add an arrow pointing from the longer root to the shorter root. B_2  G_2 

Example/exercise Show that the Dynkin diagram

F_4 : 

correspond to Coxeter matrix

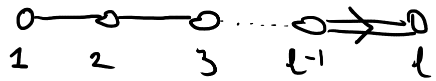
$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Observation A root system is irreducible if and only if its Coxeter graph is connected. The connected components of Δ (viewed as graph) span irreducible sub-root systems.

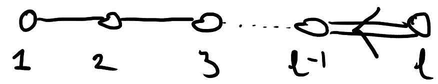
Classification theorem Assume R is irreducible. Its Dynkin diagram is one of the following:

A_ℓ $(\ell \geq 1)$:

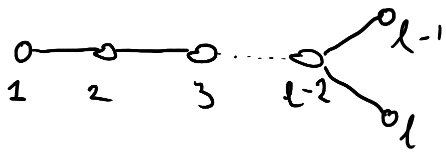
B_ℓ $(\ell \geq 2)$:



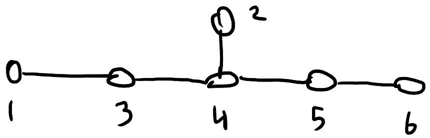
C_ℓ $(\ell \geq 3)$:



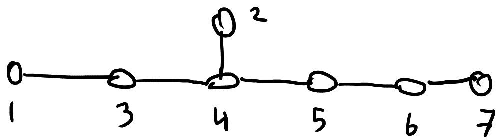
D_ℓ $(\ell \geq 4)$:



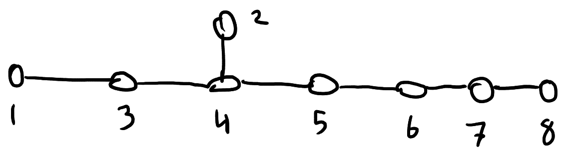
E_6:



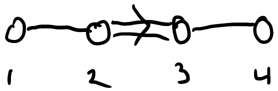
E_7:



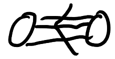
E_8:



F_4:



G_2:



__Proof__ The proof consists of many steps. The first observation, is that the Dynkin diagram is determined up to isomorphism by the Coxeter graph, except for B_ℓ, C_ℓ. But we claim that both occur, so it suffices to classify Coxeter graphs.

A linearly independent set of vectors $\{e_1, \dots, e_n\}$ of length 1 in a Euclidean space, is called admissible if for all $i \neq j$ we have

- $4(e_i, e_j)^2 \in \{0, 1, 2, 3\}$
- $(e_i, e_j) \leq 0$

By rescaling roots in Δ, we obtain an admissible set.
To every admissible set we can attach a Coxeter graph. Let's classify those.

Some observations:

(1) A subset of an admissible set is admissible, whose graph is obtained as subgraph of the Coxeter graph of the bigger set.

(2) The Coxeter graph is a tree.

It suffices to show that the number of pairs of vertices connected by at least one edge is less than n: by (1) this implies that there cannot be any cycles.

Now put $e = \sum_{i=1}^{n} e_i$. Since the e_i are linearly independent, we have $e \neq 0$.

Therefore $0 < (e,e) = n + 2 \sum_{i<j} (e_i, e_j)$.

So we have $2\sum_{i<j}(e_i, e_j) < -n$

For all $(e_i, e_j) \neq 0$ we have $(e_i, e_j) \leq 0$ and $4(e_i, e_j)^2 = 1, 2, \text{or } 3.$

Hence $2(e_i, e_j) \leq -1$, so by the above inequality there are at most $n-1$ pairs with $(e_i, e_j) \neq 0$.

(3) Every vertex can have at most three edges connected to it.

Fix a vertex ε, and let $\eta_1, \ldots, \eta_k$ be the vertices connected to ε by at least one edge. By (2) there can be no edges between η_i and η_j, so for $i \neq j$ we have $(\eta_i, \eta_j) = 0$.

Now let η_0 be a unit vector in the span of $\varepsilon, \eta_1, \ldots, \eta_k$ that is orthogonal to all $\eta_1, \ldots, \eta_k$.

Then we find $\varepsilon = \sum_{i=0}^{k} (\varepsilon, \eta_i)\, \eta_i$ and hence $1 = (\varepsilon, \varepsilon) = \sum_{i=0}^{k} (\varepsilon, \eta_i)^2$.

But then $\sum_{i=1}^{k} (\varepsilon, \eta_i)^2 < 1$, since $(\varepsilon, \eta_0) \neq 0$.

This leads to $\sum_{i=1}^{k} 4(\varepsilon, \eta_i)^2 < 4$, and $4(\varepsilon, \eta_i)^2$ is the number of edges between ε and η_i.

(4) Corollary: The only connected Coxeter graph of an admissible set that contains a triple edge is G_2.

(5) Suppose that X is admissible, and that $\{e_1, \ldots, e_k\} \subset X$ has a simple chain o—o—o…o—o as subgraph.
Put $e = \sum_{i=1}^{k} e_i$. Then $X' = (X - \{e_1, \ldots, e_k\}) \cup \{e\}$ is admissible.

It's Coxeter graph is the Coxeter graph in which the simple chain is collapsed to 1 vertex.

Linear independence of X' is clear.

For $i = 1, \ldots, k-1$ we have $2(e_i, e_{i+1}) = -1$ by assumption, and thus

$$(e,e) = k + 2\sum_{i<j} (e_i\, e_j) = k + (k-1)\cdot -1 = 1.$$

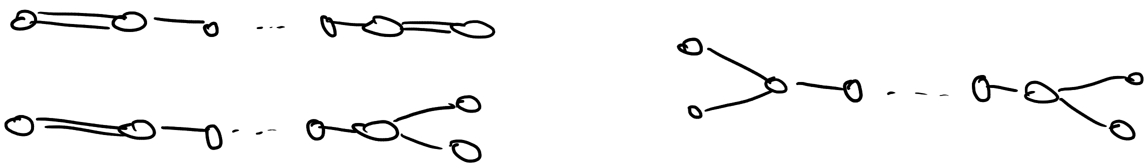
Any $\eta \in X - \{e_1, \ldots, e_k\}$ is connected to at most one e_i, by (2).

Thus we either have $(\eta, e) = 0$ or $(\eta, e) = (\eta, e_i)$ for some $1 \leq i \leq k$.

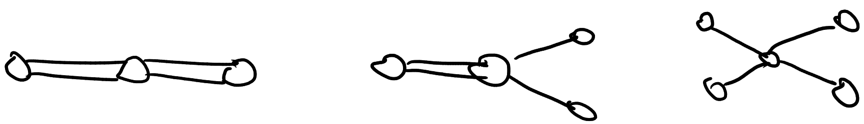
In either case $(\eta, e) \leq 0$ and $4(\eta, e)^2 = 0, 1, 2,$ or 3.

(6) Apart from G_2, there is at most one vertex with 3 edges connected to it.

Indeed, by (5) any subgraph of the form



can be contracted to

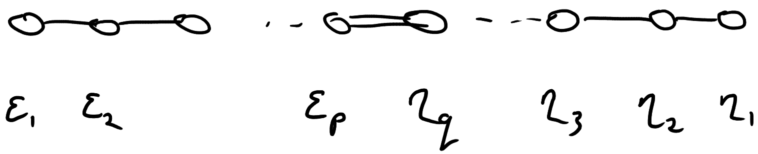


which contradicts (3).

(7) Corollary: every Coxeter graph must have one of the following forms

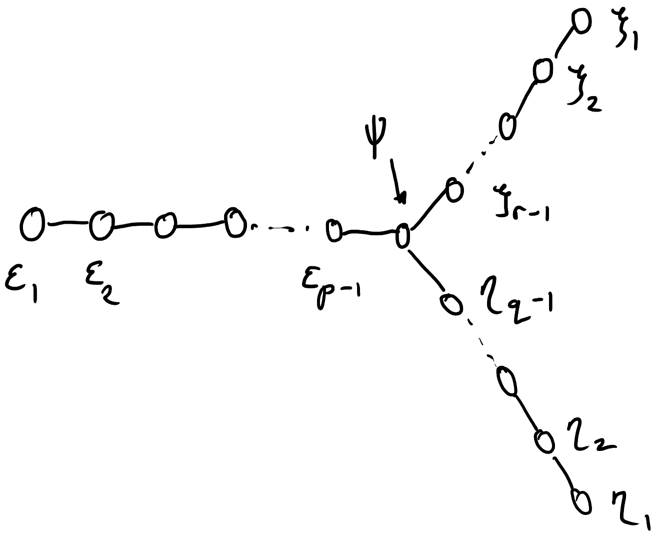
(I)

(II)



(III)

(IV)



We are now almost done. We need two more steps.

(8) The only connected Coxeter graphs of shape (II) are

F_4  and $B_\ell = C_\ell$  .

Define $\varepsilon = \sum_{i=1}^{p} i\varepsilon_i$ and $\eta = \sum_{i=1}^{q} i\eta_i$.

Our assumptions are $2(\varepsilon_i, \varepsilon_{i+1}) = -1 = 2(\eta_i, \eta_{i+1})$, $4(\varepsilon_p, \eta_q)^2 = 2$

and all other pairs are orthogonal.

Hence $(\varepsilon, \varepsilon) = \sum_{i=1}^{p} i^2 - \sum_{i=1}^{p-1} i(i+1) = \frac{p(p+1)}{2}$ and $(\eta, \eta) = \frac{q(q+1)}{2}$.

Since $4(\varepsilon_p, \eta_q)^2 = 2$ we find

$$(\varepsilon, \eta)^2 = p^2 q^2 (\varepsilon_p, \eta_q)^2 = \frac{p^2 q^2}{2}$$

By Cauchy-Schwartz we know

$$(\varepsilon,\eta)^2 < (\varepsilon,\varepsilon)(\eta,\eta)$$

or in other words

$$\frac{p^2 q^2}{2} < \frac{p(p+1)\,q(q+1)}{4}$$

Hence we find $(p-1)(q-1) < 2$.

The only options are

$$p = 2 = q \qquad (F_4)$$

$$\left.\begin{array}{l} p=1 \quad q \text{ arbitrary} \\ q=1 \quad p \text{ arbitrary} \end{array}\right\} \quad B_\ell = C_\ell$$

(9) The only connected Coxeter graphs of type IV (see (7)) are E_ℓ for $\ell = 6, 7, 8$ and D_ℓ.

Put $\varepsilon = \sum i \varepsilon_i$, $\eta = \sum i \eta_i$, and $\xi = \sum i \xi_i$. As in (8) we find

$$(\varepsilon, \varepsilon) = \frac{p(p-1)}{2}, \quad (\eta, \eta) = \frac{q(q-1)}{2} \quad \text{and} \quad (\xi, \xi) = \frac{r(r-1)}{2}.$$

Clearly, the central vertex ψ is not in the span of ε, η and ξ.

Let θ_ε, θ_η, and θ_ξ be the angles between ψ and ε, η, ξ, respectively.

With a calculation as in (3) we get

$$\cos^2 \theta_\varepsilon + \cos^2 \theta_\eta + \cos^2 \theta_\xi < 1.$$

Now we compute

$$\cos^2 \theta_\varepsilon = (\varepsilon, \psi)^2 / (\varepsilon, \varepsilon) \underbrace{(\psi, \psi)}_{=1}$$

$$= (p-1)^2 (\varepsilon_{p-1}, \psi)^2 / (\varepsilon, \varepsilon)^2$$

$$= (p-1)^2 \cdot \tfrac{1}{4} \Big/ \big(p(p-1)/2\big)$$

$$= (p-1)/2p$$

$$= \tfrac{1}{2}\left(1 - 1/p\right)$$

And similarly $\cos^2 \theta_\eta = \frac{1}{2}(1 - 1/q)$, $\cos^2 \theta_\zeta = \frac{1}{2}(1 - 1/r)$.

Thus we find $\frac{1}{2}\left(1 - 1/p + 1 - 1/q + 1 - 1/r\right) < 1$.

We can rewrite this to $1/p + 1/q + 1/r > 1$ and

without loss of generality we may assume $1/p \leq 1/q \leq 1/r$.

We also assume $1/r \leq 1/2$, because if $r = 1$ then we are in type (I) $= A_\ell$.

Hence $1/p + 1/q + 1/r > 1$ implies $3/2 \geq 3/r > 1$ and hence $r = 2$.

We are left with $1/p + 1/q > 1/2$ and $1/p \leq 1/q$, hence $2/q > 1/2$.

So we find $q = 2$ or $q = 3$. Suppose $q = 3$. Then $1/p > 1/6$.

In total we find $(p, q, r) =$

$$\begin{cases} (p, 2, 2) & D_\ell \\ (3, 3, 2) & E_6 \\ (4, 3, 2) & E_7 \\ (5, 3, 2) & E_8. \end{cases}$$

∎