Basic properties of root systems

Definition Let $R \subset E$ and $R' \subset E'$ be two root systems.

An isomorphism between (R, E) and (R', E') is a linear isomorphism $f : E \to E'$ that sends R to R', and such that $\langle f(\alpha), f(\beta) \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in R$.

If f is such an isomorphism, then $\sigma_{f(\alpha)}(f(\beta)) = f(\sigma_\alpha(\beta))$.

Exercise Use this to show that f induces an isomorphism of Weyl groups.

<u>Definition/Exercise</u> Let (R, E) be a root system. For $\alpha \in R$, write

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}$$

Check that $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$ is a root system in E.

The root system $(R^\vee, E)$ is the dual root system of (R, E).

If (R, E) is the root system coming from a Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h} < \mathfrak{g}$, then α corresponds to t_α while $\alpha^\vee$ corresponds to H_α under the Killing form identification of $\mathfrak{h}^*$ with $\mathfrak{h}^*$.

Definition The rank of (R, E) is $\dim(E)$.

In rank 1, there is only one root system:

$$A_1: \quad -\alpha \longleftarrow \;\longrightarrow \alpha$$

The Weyl group of A_1 is S_2.

In rank 2, there are 4 examples. See the exercise sheet.

However, to show that those 4 examples are exhaustive, we need to do some work.

<u>Geometric restrictions</u> Let α, β be two roots in a root system $R \subset E$.

Recall that the angle θ between α and β satisfies:

$$\cos\theta = \frac{(\alpha,\beta)}{\|\alpha\| \cdot \|\beta\|}.$$

Therefore $\langle \alpha, \beta \rangle = \dfrac{2(\alpha,\beta)}{(\beta,\beta)} = 2 \dfrac{\|\alpha\|}{\|\beta\|} \cos\theta$

and $\langle \alpha,\beta\rangle\langle\beta,\alpha\rangle = 4\cos^2\theta.$

Now we exploit $\langle \alpha,\beta\rangle \in \mathbb{Z}$ and $0 \leq \cos^2\theta \leq 1$ to find

strong restrictions on $\langle \alpha, \beta \rangle$ and θ.

We get the following table (Check this!)

	$4\cos^2\theta$	θ	$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	$\|\beta\|^2 / \|\alpha\|^2$
$\beta = \alpha$	4	0	2	2	1
$\beta \neq \pm\alpha$, $\|\beta\| \geq \|\alpha\|$	3	$\pi/6$	1	3	3
	2	$\pi/4$	1	2	2
	1	$\pi/3$	1	1	1
	0	$\pi/2$	0	0	undetermined
	1	$2\pi/3$	-1	-1	1
	2	$3\pi/4$	-1	-2	2
	3	$5\pi/6$	-1	-3	3
$\beta = -\alpha$	4	π	-2	-2	1

Lemma Let α, β be roots, $\alpha \neq \pm\beta$. If $(\alpha, \beta) > 0$ then $\alpha - \beta$ is a root. If $(\alpha, \beta) < 0$ then $\alpha + \beta$ is a root.

Proof Exercise sheet. $\blacksquare$

We continue with the assumption $\alpha \neq \pm\beta$. Consider all roots of the form

$$\beta + i\alpha \,, \qquad i \in \mathbb{Z},$$

called the α-string through β. Let $r, q \in \mathbb{Z}_{\geq 0}$ be the largest integers such that $\beta - r\alpha \in R$ and $\beta + q\alpha \in R$.

Claim
- The α-string through β is unbroken from $\beta - r\alpha$ to $\beta + q\alpha$.
- $r - q = \langle \beta, \alpha \rangle$
- Root strings have length at most 4.

Proof Suppose that $\beta + i\alpha \notin R$ for some $-r < i < q$, then there will be some $p < s$ such that

$$\beta + p\alpha \in R, \quad \beta + (p+1)\alpha \notin R, \quad \beta + (s-1)\alpha \notin R, \quad \beta + s\alpha \in R.$$

Hence we find $(\alpha, \beta + p\alpha) \geq 0$ and $(\alpha, \beta + s\alpha) \leq 0$.
But $p < s$ and $(\alpha, \alpha) > 0$, so we reach a contradiction.

Note that σ_α preserves this string, hence

$$\beta - r\alpha = \sigma_\alpha(\beta + q\alpha) = \beta + q\alpha - \frac{2(\beta + q\alpha, \alpha)}{(\alpha, \alpha)}\alpha = \beta - \langle \beta, \alpha \rangle \alpha - q\alpha$$

This shows $r - q = \langle \beta, \alpha \rangle \leq 4$