Root systems

We continue the setting from last time:

K algebraically closed field, characteristic 0

$\mathfrak{g}$ finite-dimensional semisimple Lie algebra over K

$\mathfrak{h}$ Cartan subalgebra of $\mathfrak{g}$

Today we continue our investigation of the set of roots $R \subset \mathfrak{h}^*$.

We will package the information that we get into a definition: a root system.

We have the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

into orthogonal subspaces with respect to the Killing form B.

In particular, $B|_{\mathfrak{h}}$ is nondegenerate, and we can transfer the form to $\mathfrak{h}^*$.

We denote the resulting form by $(\alpha, \beta) = B(t_\alpha, t_\beta)$

We know that R spans $\mathfrak{h}^*$, so let $\alpha_1, \ldots, \alpha_\ell$ be a basis of $\mathfrak{h}^*$

consisting of roots.

Let $\beta \in R$ be a root, and write $\beta = \sum_{i=1}^{\ell} c_i \alpha_i$, $c_i \in K$.

<u>Claim</u> $c_i \in \mathbb{Q}$

<u>Proof</u> For $j = 1, \dots, \ell$, we compute

$$(\beta, \alpha_j) = \sum_{i=1}^{\ell} c_i (\alpha_i, \alpha_j)$$

Hence

$$\underbrace{\frac{2(\beta, \alpha_j)}{(\alpha_j, \alpha_j)}}_{\in \mathbb{Z}} = \sum_{i=1}^{\ell} \underbrace{\frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}}_{\in \mathbb{Z}} \cdot c_i$$

By the results from last time we know that these coefficients are integers.

Consider the system of equations $$\frac{2(\beta, \alpha_j)}{(\alpha_j, \alpha_j)} = \sum_{i=1}^{\ell} \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \cdot X_i$$

The matrix $\left(\frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \right)_{ij}$ is nonsingular because $(\_,\_)$ is a nondegenerate form.

Hence, if a solution exists, it will be unique, and it will lie in the coefficient field of the equation, which we showed is $\mathbb{Q}$.

We know that this system of linear equations has solutions $c_i \in K$.

We conclude $c_i \in \mathbb{Q}$.

As a consequence, we see that the roots span an ℓ-dimensional $\mathbb{Q}$-vector space $\langle R \rangle \subset \mathfrak{h}^*$, denoted $E_{\mathbb{Q}}$.

In fact, $(-,-)$ restricts to a $\mathbb{Q}$-bilinear form on $E_{\mathbb{Q}}$ as we will now show. For $\lambda, \mu \in \mathfrak{h}^*$ we have

$$(\lambda, \mu) = B(t_\lambda, t_\mu) = \mathrm{Tr}(t_\lambda \circ t_\mu) = \sum_{\alpha \in R} \mathrm{Tr}\left(t_\lambda|_{\mathfrak{g}_\alpha} \circ t_\mu|_{\mathfrak{g}_\alpha}\right)$$

$$= \sum_{\alpha \in R} (\alpha, \lambda)(\alpha, \mu).$$

In particular, for $\beta \in R$, we have $(\beta, \beta) = \sum_{\alpha \in R} (\alpha, \beta)^2$.

$$(\beta,\beta) = \sum_{\alpha\in R} (\alpha,\beta)^2$$

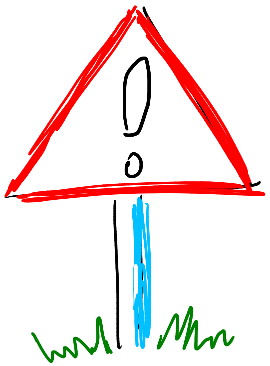
Now divide by $(\beta,\beta)^2$ to get

$$\frac{1}{(\beta,\beta)} = \sum_{\alpha\in R} \frac{(\alpha,\beta)^2}{(\beta,\beta)^2}.$$

Since $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$, we conclude that $(\beta,\beta)\in\mathbb{Q}$

and hence $(\alpha,\beta)\in\mathbb{Q}$ for all $\alpha,\beta\in R$.

Therefore $(-,-)$ is a nondegenerate rational bilinear form on $E_{\mathbb{Q}}$.



The main benefit of this reduction to

$$(-,-) : E_{\mathbb{Q}} \times E_{\mathbb{Q}} \longrightarrow \mathbb{Q}$$

is that $\mathbb{Q}$ has a notion of positivity.

<u>Theorem</u> $(-,-)$ is positive definite.

Proof: $(\beta,\beta) = \sum_{\alpha \in R} (\alpha,\beta)^2$ is a sum of squares. $\square$

Finally, we extend scalars from $\mathbb{Q}$ to $\mathbb{R}$, to obtain a real vector space E. Formally $E = E_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$.

Summary:

- E is a euclidean space, with inner product $(-,-)$
- R contains a basis of E
- $\dim_{\mathbb{R}}(E) = \ell$.

The situation can be summed up in the notion of root system.

Root system

Before defining root systems, let's talk about reflections in euclidean space.

A reflection is an automorphism that fixes pointwise a subspace of codimension 1, and that send vectors orthogonal to this hyperplane to their negative.

Exercise (i) Show that reflections are orthogonal (preserve the inner product)

(ii) For $\alpha \neq 0$, show that $\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ defines a reflection in the hyperplane $P_\alpha = \{\beta \mid (\beta,\alpha) = 0\}$.

We often abbreviate $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ by $\langle \beta, \alpha \rangle$.

A root system consists of:

- a euclidean space E
- a finite subset $R \subset E$

such that

- R spans E, and $0 \notin R$
- for $\alpha \in R$, the only multiples of α in R are $\pm\alpha$
- for $\alpha, \beta \in R$, we have $\langle \beta, \alpha \rangle \in \mathbb{Z}$.
- for $\alpha \in R$, the reflection σ_α preserves R.

The Weyl group

Let $R \subset E$ be a root system. The Weyl group W is the subgroup of $GL(E)$ generated by the reflections σ_α for $\alpha \in R$.

By the axioms of a root system, W preserves the set of roots R and since R spans E, we can identify W with a subgroup of the permutation group of R.

In particular W is finite.