Roadmap for the classification

Let $\mathfrak{g}$ be a semisimple Lie algebra

over an algebraically closed field K of characteristic 0.

Step 1 Find a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ that acts "diagonally".

- Act via semisimple endomorphisms on representations
- Elements of $\mathfrak{h}$ should "commute" $\rightsquigarrow$ $\mathfrak{h}$ must be abelian.
- $\mathfrak{h}$ should be maximal.

  Such a subalgebra is called Cartan subalgebra

Step 2 Decompose the adjoint representation

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* - \{0\}} \mathfrak{g}_\alpha$$

root spaces

For $\alpha \in \mathfrak{h}^*$, $H \in \mathfrak{h}$ and $X \in \mathfrak{g}_\alpha$ we get

$$[H, X] = \alpha(H) \cdot X.$$

Similarly $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

Set of roots $R = \{ \alpha \in \mathfrak{h}^* - \{0\} \mid \mathfrak{g}_\alpha \neq 0 \}$

Facts about roots and root spaces

- The root spaces $\mathfrak{g}_\alpha$ are 1-dimensional
- $R \subset \mathfrak{h}^*$ generates a lattice Λ_R (= free abelian grp) whose rank is equal to $\dim(\mathfrak{h})$.
- R is symmetric around the origin:

$$\alpha \in R \iff -\alpha \in R$$

We can now also decompose representations:

Let V be a finite-dimensional representation of $\mathfrak{g}$.

Then

$$V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$$

weight spaces

For $H \in \mathfrak{h}$ and $v \in V_\alpha$ we have $H(v) = \alpha(H) \cdot v$.

For $\beta \in R$ a root and $X \in \mathfrak{g}_\beta$, $X(v) \in V_{\alpha+\beta}$.

<u>Fact</u>: All the weights of V are congruent modulo Λ_R.

Step 3 $\quad \mathfrak{sl}_2$-triples

Recall from our study of $\mathfrak{sl}_3$ that we used subalgebras isomorphic to $\mathfrak{sl}_2$, so-called $\mathfrak{sl}_2$-triples.

Facts

- $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is a 1-dimensional subspace of $\mathfrak{h}$.
- $\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is isomorphic to $\mathfrak{sl}_2$.
- There is a unique element $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ that has eigenvalue ± 2 on $\mathfrak{g}_{\pm\alpha}$.

Step 4 The weight lattice

Let Λ_W be the lattice in $\mathfrak{h}^*$ consisting of $\beta \in \mathfrak{h}^*$ such that $\beta(H_\alpha) \in \mathbb{Z}$ for all α.

This lattice is called the weight lattice.

Fact all weights of all representations lie in Λ_W.

Step 5 The Weyl group

Recall that the weight of a representation of sl_3 were preserved by a natural action of S_3 generated by reflections.

For every root α consider the involution of $\mathfrak{h}^*$

$$\sigma_\alpha(\beta) = \beta - \beta(H_\alpha)\alpha.$$

The Weyl group W is the group generated by the σ_α.

__Fact__ For every representation of $\mathfrak{g}$, the set of its weights is preserved by the Weyl group W.

If β is a weight of a representation V and α a root then $\{ k \in \mathbb{Z} \mid \beta + k\alpha \text{ is a weight of } V \}$ is an uninterrupted string of even integers

$$-m, \; -m+2, \; \dots, \; 0, \; \dots, \; n-2, \; n .$$

Step 6 Order the roots

We can choose a real linear functional

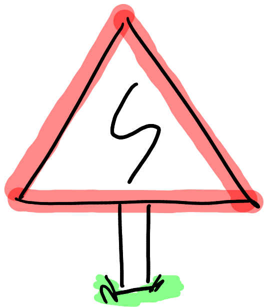
$$\ell : \Lambda_W \longrightarrow \mathbb{R}$$

that is irrational. We get a decomposition

$$R = R^+ \cup R^-$$

where $R^+ = \{ \alpha \mid \ell(\alpha) > 0 \}$ "positive roots"

$R^- = \{ \alpha \mid \ell(\alpha) < 0 \}$ "negative roots"



This choice is highly non-canonical, but we do it anyway!

Let V be a finite-dimensional representation of $\mathfrak{g}$.

<u>Definition</u> A nonzero $v \in V$ is a heighest weight vector of V if it is contained in V_β for some $\beta \in \Lambda_W$ and $\mathfrak{g}_\alpha(v) = 0$ for all positive roots α.

The weight α of a heighest weight vector is called the heighest weight or the dominant weight of V.

(All this depends very much on the choice of ordering that we made.)

<u>Proposition</u>

- □ V has a highest weight vector v
- □ The subspace generated by successive applications of $\mathfrak{g}_\beta$ for $\beta \in R^-$ is an irreducible subrepresentation.
- □ An irreducible representatiôn has a unique height weight vector up to scalars.

Step 7 The root system

Fact The Killing form is positive definite on Λ_W.

Thus we can view Λ_W as a lattice inside a Euclidean space

$$E = \Lambda_W \otimes \mathbb{R}.$$

where the scalar product comes from the Killing form.

Fact For all $\alpha, \beta \in R$, the real number

$$n_{\beta\alpha} = 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}$$

is an integer !

This means that there are only a very limited number of possible angles θ between α and β.

If we assume $\|\beta\| \geq \|\alpha\|$, then the options are:

θ	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$
$n_{\beta\alpha}$	3	2	1	0	-1	-2	-3
$\frac{\|\beta\|}{\|\alpha\|}$	$\sqrt{3}$	$\sqrt{2}$	1	?	1	$\sqrt{2}$	$\sqrt{3}$

The data $(E, \langle -,- \rangle, R)$ is called the root system of $\mathfrak{g}$.

Step 8 The Dynkin diagram

Let $\Delta \subset R^+$ be the subset of positive roots that can not be written as sum of two positive roots.

Terminology: $\alpha \in \Delta$ is called a simple or primitive positive root.

For $\alpha, \beta \in \Delta$, the angle θ will always be $\geq \pi/2$.

Now turn Δ into a graph, the Dynkin diagram:

assume $\|\beta\| \geq \|\alpha\|$ and add $|n_{\beta\alpha}|$ lines between β and α.
If $\|\beta\| > \|\alpha\|$, then add an arrow pointing from β to α.

<u>Fact</u> The isomorphism type of the Dynkin diagram does not depend on the choice of ordering of roots that we made.

<u>Theorem</u>

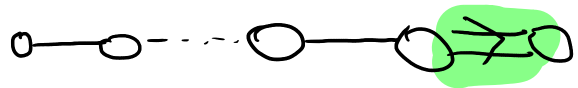
- $\square$ Two semisimple Lie algebras are isomorphic if and only if their Dynkin diagrams are isomorphic.
- $\square$ The simple summands of a Lie algebra correspond to the connected components of the Dynkin diagram:

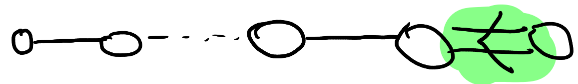
$$\mathrm{Dyn}(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \cong \mathrm{Dyn}(\mathfrak{g}_1) \sqcup \mathrm{Dyn}(\mathfrak{g}_2)$$

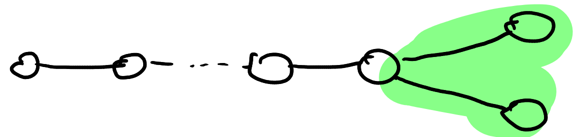
The connected Dynkin diagrams are

"Classical"

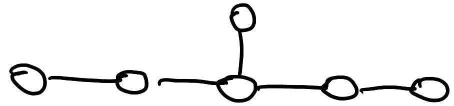
$\mathfrak{sl}_{n+1}$ A_n 

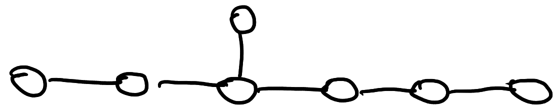
$\mathfrak{so}_{2n+1}$ B_n 

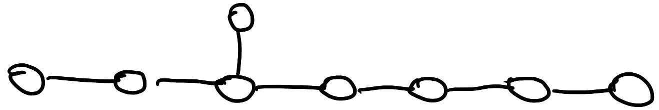
$\mathfrak{sp}_{2n}$ C_n 

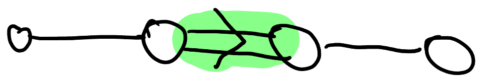
$\mathfrak{so}_{2n}$ D_n 

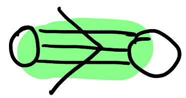
"Exceptional"

E_6 

E_7 

E_8 

F_4 

G_2 

The subscript in the labels $A_n, B_n, \ldots, G_2$ is the number of nodes in the diagram.

Step 9 Classify the irreducible representations

Recall that the Weyl group W acts on the weight lattice Λ_W.

For every weight α there exists a finite-dimensional irreducible representation Γ_α (unique up to isomorphism) and an ordering of the roots such that α is the highest weight of Γ_α.

Furthermore, $\Gamma_\alpha \cong \Gamma_\beta$ if and only if α and β are in the same orbit under the action of the Weyl group W.

Step 10 The dimension of the weight spaces

For the irreducible representation T_α, determine the dimension of the weight space $(T_\alpha)_\beta$.

This is done by the Weyl character formula which we will hopefully see at the end of the course.