Nilpotent and solvable Lie algebras

We continue the list of definitions.

The main properties that we will see in this lecture:

- ☐ nilpotent
- ☐ solvable

Next time:

- ☐ simple
- ☐ semisimple

Nilpotent Lie algebras

A Lie algebra $\mathfrak{g}$ is nilpotent if there exists a decreasing chain of ideals

$$\mathfrak{g} = \mathfrak{g}_0 \supset \cdots \supset \mathfrak{g}_k = 0$$

such that $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$ for all i

Exercise The following are equivalent:

(i) $\mathfrak{g}$ is nilpotent

(ii) $\mathcal{D}_k \mathfrak{g} = 0$ for some k

(iii) $\mathcal{C}_k \mathfrak{g} = \cancel{0}\ \mathfrak{g}$ for some k

(This mistake unfortunately slipped into the video recording.)

Corollary If $\mathfrak{g}$ is nilpotent and $\mathfrak{g} \neq 0$ then the centre $Z(\mathfrak{g})$ is not 0.

Solvable Lie algebras

A Lie algebra is solvable if there exists a decreasing chain of ideals

$$\mathfrak{g} = \mathfrak{g}_0 \supset \cdots \supset \mathfrak{g}_k = 0$$

such that $\mathfrak{g}_i / \mathfrak{g}_{i+1}$ is abelian.

<u>Exercise</u> The following are equivalent

(i) $\mathfrak{g}$ is solvable

(ii) $\mathcal{D}^k \mathfrak{g} = 0$ for some k

(iii) There exists a decreasing chain of subalgebras

$$\mathfrak{g} = \mathfrak{g}_0 \supset \cdots \supset \mathfrak{g}_k = 0$$

such that $\mathfrak{g}_{i+1}$ is an ideal in $\mathfrak{g}_i$ and

$\mathfrak{g}_i / \mathfrak{g}_{i+1}$ is abelian.

Exercise: A sub/quotient Lie algebra of a nilpotent (resp. solvable) Lie algebra is itself nilpotent (resp. solvable).

Exercise: Suppose that $\mathfrak{h} \subset \mathfrak{g}$ is an ideal. Then $\mathfrak{g}$ is solvable if and only if $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are.

Corollary: The sum of two solvable ideals is solvable.

Exercises

(i) Every nilpotent Lie algebra is solvable.

(ii) The Lie subalgebra $\mathfrak{n}_k(\mathbb{R}) \subset \mathrm{Mat}_{k\times k}(\mathbb{R})$ of strictly uppertriangular $k \times k$-matrices is nilpotent.

so with 0 on the diagonal

(iii) The Lie subalgebra $\mathfrak{b}_k(\mathbb{R}) \subset \mathrm{Mat}_{k\times k}(\mathbb{R})$ of upper triangular $k\times k$ matrices is solvable, but not nilpotent (for $k \geq 2$).