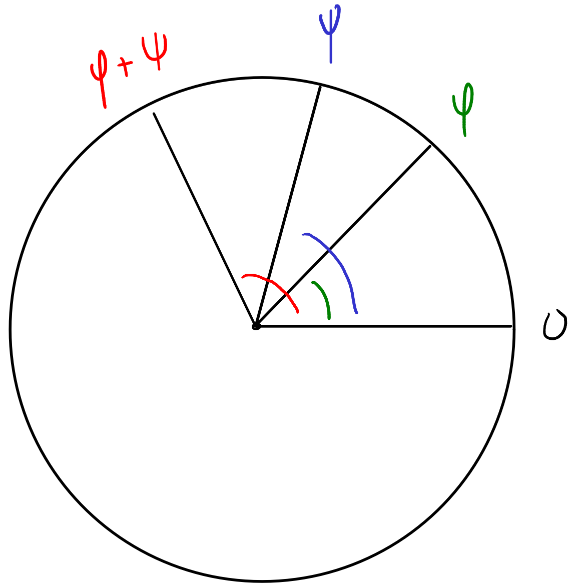
The geometric motivation

Lie groups are a class of groups that we encounter often in mathematics and they mix algebra and geometry.

Examples

- $\mathbb{R}^n$ is an additive Lie group
- S^1 (the circle group)
- $GL_n(\mathbb{R})$ is an example of a non-commutative Lie group
- $O_n(\mathbb{R})$ [orthogonal matrices]
- $SL_n(\mathbb{R})$, $SO_n(\mathbb{R})$



Formally

A Lie group is a group and a manifold such that

$$m : G \times G \longrightarrow G$$ (multiplication)

and

$$\iota : G \longrightarrow G$$ (inversion)

are smooth

Tangent spaces and derivatives (recap)

Let M and N be manifolds

and $f : M \to N$ a smooth map.

Let $x \in M$ be a point. Recall:

$$T_x M \xrightarrow{(df)_x} T_{f(x)} N$$

If $f : G \longrightarrow H$ is a morphism of Lie groups

then we get $T_e G \xrightarrow{(df)_e} T_e H$.

<u>Fact</u> f is completely determined by $(df)_e$.

Q: Which maps $T_e G \longrightarrow T_e H$ come from

Lie group morphisms $G \longrightarrow H$?

$\boxed{\text{Spoiler alert!}}$

$T_e G$ is naturally a Lie algebra

$(df)_e$ a Lie algebra morphism

This puts conditions on $(df)_e$.

Q: Which maps $T_eG \longrightarrow T_eH$ come from Lie group morphisms?

For $g \in G$, let $m_g : G \to G$, $h \longmapsto gh$ denote left-multiplication by g.

A smooth function $G \xrightarrow{f} H$ is a morphism iff

$$\begin{tikzcd}
h \in G \arrow[r, "f"] \arrow[d, "m_g"'] & H \arrow[d, "m_{f(g)}"] \\
G \arrow[r, "f"] & H
\end{tikzcd}$$

commute for all $g \in G$.

$$f(g)\cdot f(h) = m_{f(g)}(f(h)) = f(m_g(h)) = f(g \cdot h)$$

Little problem: $m_g(e) \neq e$ if $g \neq e$.

So we don't stay inside $T_e G$.

"Solution": consider $\psi_g : G \longrightarrow G$, $h \longmapsto ghg^{-1}$ — conjugation by g.

Recall:

$$\begin{array}{ccccc} Z(G) & \hookrightarrow & G & \longrightarrow & \mathrm{Aut}(G) \\ & & g & \longmapsto & \psi_g \end{array}$$

centre of G

$$\begin{aligned} Z(G) \hookrightarrow\ & G \longrightarrow \operatorname{Aut}(G) \\ & g \longmapsto \psi_g \end{aligned}$$

Fortunately, many Lie groups have a relatively small centre.

In any case,

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\psi_g"'] & H \arrow[d, "\psi_{f(g)}"] \\ G \arrow[r, "f"] & H \end{tikzcd}$$

must commute, if f is a morphism of Lie groups.

$$\Psi_g : G \longrightarrow G$$

$$\rightsquigarrow$$

$$\begin{tikzcd}
T_e G \arrow[r, "\mathrm{Ad}(g)", "\sim"'] \arrow[d, "(df)_e"'] & T_e G \arrow[d, "(df)_e"] \\
T_e H \arrow[r, "\mathrm{Ad}(f(g))"'] & T_e H
\end{tikzcd}$$

$$\mathrm{Ad}(g) : T_e G \xrightarrow{\sim} T_e G$$

$$\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(T_e G)$$

$$\rightsquigarrow$$

$$\mathrm{ad} : T_e G \longrightarrow \mathrm{End}(T_e G)$$

$\forall X \in T_e G$

$$\begin{tikzcd}
T_e G \arrow[r, "\mathrm{ad}(X)"] \arrow[d, "(df)_e"'] & T_e G \arrow[d, "(df)_e"] \\
T_e H \arrow[r, "\mathrm{ad}((df)_e(X))"'] & T_e H
\end{tikzcd}$$

__Definition__ (Lie bracket)

For $X, Y \in T_eG$,

$$[X,Y] = ad(X)(Y)$$

$\forall X \in T_eG$

$$\begin{tikzcd} T_eG \arrow[r, "ad(X)"] \arrow[d, "(df)_e"] & T_eG \arrow[d, "(df)_e"] \\ T_eH \arrow[r, "ad((df)_e(X))"'] & T_eH \end{tikzcd}$$

$\forall X, Y$

$$(df)_e(ad(X)\,Y) = ad((df)_e(X))((df)_e\,Y)$$

$$\Updownarrow$$

$$(df)_e[X, Y] = [(df)_e(X), (df)_e\,Y]$$

$(df)_e$ commutes $[-,-]$!

Fact 1

For $G = GL_n(\mathbb{R})$, $T_eG = \mathrm{Mat}_{n\times n}(\mathbb{R})$

and the Lie bracket is the usual

commutator bracket of matrices

Exercise: Check this, if you have some

experience in differential geometry.

Consequence of fact 1

The bracket is bilinear,

alternating: $[x,x]=0$ and $[x,y]=-[y,x]$

and satisfies the Jacobi identity:

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0.$$

Definition A Lie algebra is a vector space endowed with an alternating bilinear operator that satisfies the Jacobi identity.

Fact 2

Let G and H be Lie groups and suppose that G is connected and simply connected.

Then every linear map $T_e G \to T_e H$ that preserves the Lie bracket is the differential of a morphism of Lie groups $G \to H$.