The algebraic motivation

Let A be an algebra.

Recall the commutator bracket:

$$[x,y] = xy - yx$$

Q: What are the formal properties of this bracket?

Observations

$$\left.\begin{aligned} [x,x] &= 0 \\ [x,y] &= -[y,x] \end{aligned}\right\}$$ alternating

$$[x,y] = \underbrace{xy - yx}_{\text{linear in both } x \text{ and } y} \rightsquigarrow \text{bilinear}$$

Experiment

Let A and B be vector spaces

(over some field K)

with alternating bilinear operators

$$[-,-] : A \times A \longrightarrow A$$

$$[-,-] : B \times B \longrightarrow B$$

Experiment (cntd)

A linear map $f : A \to B$

is a "bracket morphism" if

$$f[x, y] = [fx, fy].$$

Experiment (cntd)

This has the commutator bracket!

Consider $A \longrightarrow \mathrm{End}(A)$

$x \longmapsto [x, -]$

Q: Is this a

"bracket morphism"?

Experiment (cntd)

$$A \longrightarrow \mathrm{End}(A)$$
$$x \longmapsto [x, \_\,]$$

$$[x,y] \longmapsto [[x,y], \_\,]$$

$$\overset{???}{=}$$

$$[[x,-],[y,-]]$$

$$=$$

$$[x,[y,-]] - [y,[x,-]]$$

Definition of bracket commutator on $\mathrm{End}(A)$.

$$[f,g] = fg - gf$$

Conclusion: For all $x,y,z \in A$ we need

$$[[x,y],z] = [x,[y,z]] - [y,[x,z]].$$

Experiment (cntd)

$$[[x,y],z] = [x,[y,z]] - [y,[x,z]]$$

$$= -[[y,z],x] + [[x,z],y]$$

$$= -[[y,z],x] - [[z,x],y]$$

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$$

Jacobi Identity

Definition

A Lie algebra is a vector space with an alternating bilinear operator $[-,-]$ (the "Lie bracket") that satisfies the Jacobi identity.

A morphism of Lie algebras is a bracket morphism.

Crucial exercise

Let A be a K-algebra.

Check that the commutator bracket

$$[x,y] = xy - yx$$

satisfies the Jacobi identity