Lie's theorem

Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a solvable Lie algebra over an algebraically closed field.

Then there exists a nonzero vector $v \in V$ that is an eigenvector of X for all $X \in \mathfrak{g}$.

Similar to Engel's theorem: in that case, eigenvalues all $= 0$

Proof also follows a similar pattern.

<u>Claim 1</u>: There is an ideal $\mathfrak{h}$ of codimension 1.

<u>Proof of claim</u> Since $\mathfrak{g}$ is solvable, $D\mathfrak{g} \neq \mathfrak{g}$.

Hence $\mathfrak{g}/D\mathfrak{g}$ is an abelian Lie algebra of dimension > 0.

Every subspace of $\mathfrak{g}/D\mathfrak{g}$ is an ideal.

We are done by taking the inverse image of any codimension 1 subspace under $\mathfrak{g} \to \mathfrak{g}/D\mathfrak{g}$. $\square$

Since $\dim \mathfrak{h} < \dim \mathfrak{g}$, we may assume by induction that there exists a nonzero $v_0 \in V$, that is an eigenvector for all $X \in \mathfrak{h}$.

Let $\lambda(X)$ denote the corresponding eigenvalue, so that

$$X(v_0) = \lambda(X) \cdot v_0 \quad \text{for all } X \in \mathfrak{h}.$$

Now consider

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As before: pick $Y \in \mathfrak{g}$, $Y \notin \mathfrak{h}$.

<u>Claim 2</u> The subspace W is stable under Y.

Since we are working over an algebraically closed field this implies that there is some nonzero $w \in W$ that is an eigenvector of Y, hence eigenvector of all $X \in \mathfrak{g}$. $\square$

The proof of claim 2 follows from:

<u>Lemma</u> Let $\mathfrak{h} \subset \mathfrak{g}$ be an ideal of a Lie algebra over K, V a representation of $\mathfrak{g}$, and $\lambda : \mathfrak{h} \longrightarrow K$ a linear function.

Write $W = \{ v \in V \mid X(v) = \lambda(X) \cdot v \text{ for all } X \in \mathfrak{h} \}$.

Then $Y(W) \subset W$ for all $Y \in \mathfrak{g}$.

<u>Proof</u> Pick $w \in W$. Need to show:

$$X(Y(w)) = \lambda(X) \cdot Y(w).$$

Proof: Pick $w \in W$. Need to show: $X(Y(w)) = \lambda(X) \cdot Y(w)$.

Note that $X(Y(w)) = \underbrace{Y(X(w))}_{Y(\lambda(X)\cdot w) = \lambda(X)\cdot Y(w)} + [X,Y](w)$

So, need to show $[X,Y](w) = 0$.

Consider $U = \langle w, Y(w), Y^2(w), \ldots, Y^k(w), \ldots \rangle$.

Clearly $Y(U) \subset U$.

Claim Fix $X \in \mathfrak{h}$. Then $X(Y^k(w))$ is of the form

$$\lambda(X) \cdot Y^k(w) + (\text{lin. combi. of } Y^{k-1}(w), \ldots, Y(w), w)$$

In other words: $X(U) \subset U$, and on the basis $w, Y(w), Y^2(w), \ldots$ of U, X is upper triangular with $\lambda(X)$ on all diagonal entries.

Proof of the claim By induction on k and:

$$X(Y^k(w)) = Y(X(Y^{k-1}(w))) + [X,Y](Y^{k-1}(w))$$

$\square$

Now return to the proof of the lemma.

We have $w \in W$ and wanted to show $Y(w) \in W$, i.e.,

$$X(Y(w)) = \lambda(X) \cdot Y(w).$$

Since $$X(Y(w)) = \underset{\lambda(X) \cdot Y(w)}{\underset{\shortparallel}{Y(X(w))}} + [X, Y](w)$$

we want to show $[X, Y](w) = 0$.

Note $[X, Y] \in \mathfrak{h}$, so $[X, Y](w) = \lambda([X, Y]) \cdot w$

So showing $\lambda([X, Y]) = 0$ is enough.

By the claim: $\operatorname{Tr}\left([X,Y]|_U\right) = \lambda([X,Y]) \cdot \dim(U)$.

But X and Y preserve U, so

$$[X,Y]|_U = [X|_U, Y|_U],$$

and commutators have trace 0.

So $0 = \dim(U) \cdot \lambda[X,Y]$ and since $0 \neq w \in U$,

we see that $\dim(U) \neq 0$, so $\lambda([X,Y]) = 0$ $\square$