The Killing form

- ☐ A bilinear form on a Lie algebra
- ☐ Cartan's criterion for solvability / semisimplicity
- ☐ Every semisimple Lie algebra is a sum of simple ones

Let $\mathfrak{g}$ be a Lie algebra and

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$$

a representation of $\mathfrak{g}$.

The Killing form of V is a bilinear form on $\mathfrak{g}$

$$B_V(X,Y) = \mathrm{Tr}\left(\rho(X) \circ \rho(Y)\right)$$

The Killing form of $\mathfrak{g}$ is

$$B(X,Y) = B_{\mathfrak{g}}(X,Y) = \mathrm{Tr}\left(\mathrm{ad}(X) \circ \mathrm{ad}(Y)\right)$$

Exercise Show for all $X, Y, Z \in \mathfrak{g}$ that

$$B_V([X,Y], Z) = B_V(X, [Y,Z]).$$

Exercise (i) If $\mathfrak{g}$ is solvable, then $\mathfrak{g}$ acts on $\mathfrak{g}$ via upper-triangular matrices (use Lie's theorem)

(ii) Hence $\mathcal{D}\mathfrak{g}$ acts via strictly upper-triangular matrices

(iii) Conclude that $B_V(X, Y) = 0$ for all $X \in \mathfrak{g}$, $Y \in \mathcal{D}\mathfrak{g}$.

Cartan's criterion Let $\mathfrak{g}$ be a Lie algebra.

(i) $\mathfrak{g}$ is solvable if and only if $B(\mathfrak{g}, D\mathfrak{g}) = 0$.

(ii) If $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$ and $B_V(X,Y) = 0$ for all $X, Y \in \mathfrak{g}$, then $\mathfrak{g}$ is solvable.

(iii) $\mathfrak{g}$ is semisimple if and only if its Killing form B is nondegenerate.

<u>(ii) $\Longrightarrow$ (iii)</u> Consider the ideal

$$\mathfrak{s} = \{ X \in \mathfrak{g} \mid B(X,Y) = 0 \text{ for all } Y \in \mathfrak{g} \}$$

By (ii), we see that $\mathrm{ad}(\mathfrak{s})$ is solvable, hence $\mathfrak{s}$ is solvable.

If $\mathfrak{g}$ is semisimple, then $\mathfrak{s} = 0$, hence B is nondegenerate.

Conversely, suppose $\mathfrak{a} \subset \mathfrak{g}$ is an abelian ideal.

Let $X \in \mathfrak{g}$, $Y \in \mathfrak{a}$, $A = \mathrm{ad}(X) \circ \mathrm{ad}(Y)$. $\quad \mathfrak{g} \xrightarrow{A} \mathfrak{a} \xrightarrow{A} 0$

So $B(X,Y) = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) = 0$. This means $\mathfrak{a} \subset \mathfrak{s}$.

If B is nondegenerate, then $\mathfrak{a} \subset \mathfrak{s} = 0$. So $\mathfrak{g}$ is semisimple. $\square$