RECAP: JORDAN DECOMPOSITION

0.1. The goal of this note is to recapitulate the notion of Jordan decomposition, and prove that it exists and is unique.

Let K be a field, let V be a finite-dimensional vector space over K, and let $f\colon V \to V$ be an endomorphism.

0.2. **Definition.** The endomorphism f is called *semisimple* if every f-invariant subspace $W \subset V$ has a complementary f-invariant subspace.

0.3. **Remark.** Equivalent ways of characterising semisimple endomorphisms are:

(i) The pair (V, f) is a semisimple $K[x]$-module.
(ii) The minimum polynomial of f is a product of distinct irreducible polynomials.
(iii) There is a basis of V with respect to which the endomorphism f corresponds to a diagonal matrix.

0.4. **Definition.** A *Jordan decomposition* of f consists of two endomorphisms s and n of the vector space V, such that:

(i) s is semisimple, and n is nilpotent,
(ii) $f = s + n$,
(iii) s and n commute: $s \circ n = n \circ s$.

0.5. **Lemma.** *Assume that $f = s + n$ is a Jordan decomposition of f. Then f commutes with s and n.*

0.6. From now on, we assume that K is algebraically closed.

0.7. **Theorem.** *Assume that $f = s+n$ and $f = s'+n'$ are two Jordan decompositions of f, with s and s' semisimple, and n and n' nilpotent endomorphisms. Then $s = s'$ and $n = n'$.*

Proof. Let λ be an eigenvalue of f. Consider the s-eigenspace $V_\lambda = \ker(f - \lambda I)$. Then V_λ is n-invariant, since for $v \in V_\lambda$ we have

$$s(n(v)) = n(s(v)) = n(\lambda v) = \lambda n(v)$$

and hence $n(v) \in V_\lambda$.

We conclude that V_λ is f-invariant, since $f = s + n$.

Because s is semisimple, we know that V is a direct sum of the eigenspaces of s. We claim that these eigenspaces are also the generalized eigenspaces of f. Indeed, on V_λ, the endomorphism s acts as multiplication by λ. Additionally, $f - s$ is nilpotent, say $(f - s)^m = 0$ for some $m > 0$. This means that $V_\lambda \subset \ker(f - \lambda I)^m$. In particular V_λ is contained in the generalized λ-eigenspace of f.

Now consider the decomposition of V into a direct sum of eigenspaces of s

$$V = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_k}$$

and the decomposition of V into a direct sum of generalized eigenspaces of f

$$V = W_{\mu_1} \oplus \cdots \oplus W_{\mu_l}.$$

By what we have argued before, we see that for each λ_i, there exists a μ_j such that $V_{\lambda_i} \subset W_{\mu_j}$, and in this case $\lambda_i = \mu_j$. For dimension reasons, we conclude that $V_{\lambda_i} = W_{\mu_j}$.

Hence the decompositions $V = \bigoplus_{\lambda_i} V_{\lambda_i}$ and $V = \bigoplus_{\mu_j} W_{\mu_j}$ are the same, up to a reordering of the terms.

Now we are almost done: since the subspaces W_{μ_j} are uniquely determined by f, and s acts on them by multiplication with μ_j, we see that s is uniquely determined by f.

Hence $s = s'$, and therefore $n = n'$. □

The uniqueness proof also suggests how to prove the existence of Jordan decompositions.

0.8. **Theorem.** *There exists a Jordan decomposition of f.*

Proof. As in the previous proof, let

$$V = \bigoplus_{\mu} W_\mu$$

be a decomposition of V into generalized eigenspaces of f.

Define the endomorphism s of V via: $s(v) = \mu \cdot v$ for all $v \in W_\mu$. By construction, s is semisimple. (If you want, pick a basis for each W_μ, then s is clearly a diagonal matrix with respect to this basis.)

It is similarly clear, that $n = f - s$ is a nilpotent endomorphism of V. We point out that n preserves the decomposition $V = \bigoplus_\mu W_\mu$.

It remains to check that s and n commute. However, it suffices to check this on each subspace W_μ. On such a subspace, we see that

$$n(s(v) = n(\lambda v) = \lambda n(v) = s(n(v)).$$

(Here we use that n preserves the subspace W_μ.) □

0.9. **Remark.** The two Theorems 0.8 and 0.7 together justify speaking of "the" Jordan decomposition of f.

0.10. **Proposition.** *Let $s + n$ be the Jordan decomposition of f. Then there exist polynomials $P, Q \in K[X]$ that have no constant coefficients, and such that $P(f) = s$ and $Q(f) = n$.*

Proof. It suffices to show the existence of P, because then $Q = X - P$ will evaluation on f as follows:

$$Q(f) = X(f) - P(f) = f - s = n.$$

If P has no constant coefficient, then neither does Q.

Once again, we consider the generalized eigenspace decomposition of f:

$$V = \bigoplus_{\mu} W_\mu.$$

Writing $d_\mu = \dim W_\mu$, we see that the characteristic polynomial χ_f of f satisfies

$$\chi_f = \prod_{\mu} (X - \mu)^{d_\mu}.$$

We now apply the Chinese remainder theorem to the ring $K[X]$, to find a polynomial P that satisfies:

$$P \equiv 0 \pmod{X}, \quad P \equiv \mu \pmod{(X - \mu)^{d_\mu}} \quad \text{(for all } \mu).$$

The condition $P \equiv 0 \pmod{X}$ ensures that P does not have a constant coefficient.

The other conditions mean that for every μ, we can write $P - \mu = G \cdot (X - \mu)^{d_\mu}$, for some $G \in K[X]$. Hence the action of $P(f) - \mu$ on $v \in W_\mu$ is the same as $G(f) \cdot (f - \mu)^{d_\mu}$, which acts as multiplication by 0 on W_μ.

We conclude that $P(f) = s$, which finishes the proof. □