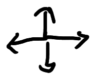
# Irreducible root systems

A root system is irreducible if it is nonempty, and if

$R = R_1 \cup R_2$ with $R_1 \perp R_2$ then $R_1 = \emptyset$ or $R_2 = \emptyset$.

Examples: $A_1$, $A_2$, $B_2$, and $G_2$ are irreducible.

$A_1 \times A_1$ is reducible :  It consists of two orthogonal copies of $A_1$.

Claim One can check irreducibility on a base $\Delta$ of $R$.

Indeed, if $R_1 \perp R_2$, then take $\Delta_i = \Delta \cap R_i$ which gives $\Delta_1 \perp \Delta_2$.

Now suppose that $\Delta = \Delta_1 \cup \Delta_2$ and $\Delta_1 \perp \Delta_2$.

Let $\alpha$ be a root. Then there is some $\sigma \in \mathcal{W}$ such that $\sigma(\alpha) \in \Delta$.

Let $R_i$ be the subset of roots conjugate to a root in $\Delta_i$.

Certainly $R = R_1 \cup R_2$. Now observe that $(\alpha, \beta) = 0$ implies $\sigma_\alpha \sigma_\beta = \sigma_\beta \sigma_\alpha$.

Similarly, if $\alpha \perp \Delta_i$ then $\sigma_\alpha$ preserves the subspace $E_i \subset E$ spanned by $\Delta_i$.

So if $\sigma(\alpha) \in \Delta_i$ and $\sigma = \sigma_{\alpha_1} \cdot \sigma_{\alpha_2} \cdots \sigma_{\alpha_t}$ with $\alpha_j \in \Delta$,

then we can assume $\alpha_j \in \Delta_i$. Hence $\alpha \in E_i$.

Clearly $E_1 \perp E_2$ and hence $R_1 \perp R_2$. This proves the claim.

Let $\Delta$ be a base of $R$. If $\beta \in R$ is $\sum_{\alpha\in\Delta} k_\alpha \cdot \alpha$, recall that

$$ht(\beta) = \sum_{\alpha\in\Delta} k_\alpha \quad \text{is the height of } \beta.$$

For the remainder of this lecture, assume $R$ is irreducible.

<u>Lemma</u> There is a unique maximal root of $R$ relative to the ordering $<$. We have $(\beta, \alpha) \geq 0$ for all $\alpha \in \Delta$, $ht\, \alpha < ht\, \beta$ for all $\alpha \neq \beta$, $\alpha \in R$ and if $\beta = \sum_{\alpha\in\Delta} k_\alpha \cdot \alpha$, then $k_\alpha > 0$ for all $\alpha \in \Delta$.

<u>Proof</u> Step 1. Let $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$ be maximal. Put

$$\Delta_1 = \{\alpha \in \Delta \mid k_\alpha > 0\}, \qquad \Delta_2 = \{\alpha \in \Delta \mid k_\alpha = 0\}.$$

Clearly $\Delta_1 \neq 0$. Suppose $\Delta_2 \neq 0$. Then there must be $\alpha' \in \Delta_1$ and $\alpha \in \Delta_2$

such that $(\alpha', \alpha) \neq 0$ by irreducibility of $R$.

Now recall that $(\gamma, \gamma') \leq 0$ for simple roots $\gamma \neq \gamma'$.

Hence

- $(\alpha, \beta) \leq 0$ since $\alpha$ is simple and $\beta$ positive
- $(\alpha, \alpha') < 0$ since $\alpha$ and $\alpha'$ are simple, but not orthogonal
- $(\alpha, \beta) < 0$ combine the two points above.

But then $\beta + \alpha$ is a root, which contradicts maximality.

So $\Delta_2 = \phi$, $k_\alpha > 0$ for all $\alpha \in \Delta$, and $(\alpha, \beta) \geq 0$ for all $\alpha \in \Delta$.

Step 2. Suppose $\beta' = \sum_{\alpha \in \Delta} k'_\alpha$ is another maximal root.

Let $\alpha \in \Delta$ be such that $(\alpha, \beta) > 0$, which exists since $\Delta$ spans $E$ and $\beta \neq 0$.

Since $k'_\alpha > 0$, we find $(\beta', \beta) > 0$.

This means that either $\beta = \beta'$ or $\beta - \beta'$ is a root.

The latter contradicts maximality of $\beta$ or $\beta'$, so we conclude $\beta = \beta'$. ∎

**Lemma** $W$ acts irreducibly on $E$.

In particular, the $W$-orbit of a root $\alpha$ spans $E$.

**Proof** Let $E'$ be a nonzero $W$-invariant subspace of $E$.
The orthogonal complement $E''$ is also $W$-invariant.
By one of the exercises, for all $\alpha$ we have

$$\alpha \in E' \quad \text{or} \quad E' \subset P_\alpha .$$

Hence $\alpha \notin E'$ implies $\alpha \in E''$.

This means that $(E' \cap R)$ and $(E'' \cap R)$ partition the roots into orthogonal subsets.

By irreducibility we conclude $E'' \cap R = \emptyset$, hence $E'' = 0$.

**Lemma** At most two root lengths occur in $\Delta$.

All roots of a given length are conjugate under $\mathcal{W}$.

**Proof** Let $\alpha, \beta$ be two roots. There exists a $\sigma \in \mathcal{W}$ such that $\sigma(\alpha)$ is not orthogonal to $\beta$, since $\mathcal{W} \cdot \alpha$ spans $E$.

Now $\|\sigma(\alpha)\| = \|\alpha\|$, and $\dfrac{\|\sigma(\alpha)\|^2}{\|\beta\|^2}$ must be one of $\frac{1}{3}, \frac{1}{2}, 1, 2, 3$.

If there were $> 3$ different lengths, then $\frac{2}{3}$ would occur, which is impossible.

Now suppose that $\alpha' = \sigma(\alpha)$ has the same length as $\beta$.

By the table with angles and Cartan integers, we find $\langle \alpha', \beta \rangle = \langle \beta, \alpha' \rangle = \pm 1$

After possibly replacing $\beta$ by $-\beta$, we get $\langle \alpha', \beta \rangle = 1$.

Then $(\sigma_\alpha \sigma_\beta \sigma_\alpha)(\beta) = \alpha$. (Check this.)

If $R$ has two different root lengths, then we naturally partition them into long roots and short roots.
If all roots have the same length, then we call all of them long.

<u>Lemma</u> The maximal root $\beta \in R$ is long.

<u>Proof</u> Let $\alpha$ be an arbitrary root. Replace $\alpha$ by a $W$-conjugate lying in the fundamental Weyl chamber of $\Delta$.

Since $\beta - \alpha > 0$, we find $(x, \beta - \alpha) \geq 0$ for all $x \in \overline{\mathcal{C}(\Delta)}$.

For $x = \beta$, this gives $(\beta, \beta) \geq (\beta, \alpha)$. For $x = \alpha$, we get $(\alpha, \beta) \geq (\alpha, \alpha)$.
We conclude $(\beta, \beta) \geq (\alpha, \alpha)$, which shows that $\beta$ is a long root. ∎