Freudenthal's multiplicity formula

Let $\mathfrak{g}$, $\mathfrak{h} \subset \mathfrak{g}$, R be as usual. Let λ be a dominant weight of R and let $V = V_\lambda$ be the irreducible representation of hi hest weight λ.

Last time: V is finite-dimensional

As usual: $$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$$

This time: If μ is a weight of V,

can we calculate the dimension of the weight space V_μ?

Notation Write n_μ for $\dim V_\mu$.

Let $\delta = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ be half the sum of the positive roots.

Write $c(\mu)$ for $\|\lambda + \delta\|^2 - \|\mu + \delta\|^2 = (\lambda+\delta, \lambda+\delta) - (\mu+\delta, \mu+\delta)$

Theorem (Freudenthal)

$$c(\mu) \cdot n_\mu = 2 \sum_{\alpha \in R^+} \sum_{k \geq 1} (\mu + k\alpha, \alpha)\, n_{\mu + k\alpha}$$

Corollary We can recursively compute n_μ, using that $n_\lambda = 1$.

The rest of this lecture is dedicated to proving the theorem.

<u>Casimir operator</u> Recall that we defined the Casimir operator.

Pick any basis $U_1, \dots, U_r$ of $\mathfrak{g}$ and let $U_1', \dots, U_r'$ be the dual basis with respect to the Killing form.

Then, for $v \in V$ we have

$$C(v) = \sum U_i(U_i'(v)).$$

Key properties (see the lecture on Casimir operators):

- $C(v)$ does not depend on the chosen basis.
- $C(X(v)) = X(C(v))$ for all $X \in \mathfrak{g}$, $v \in V$.

Step 1 Pick a useful basis:

For every $\alpha \in R$, let $(X_\alpha, H_\alpha, Y_\alpha)$ be an $\mathfrak{sl}_2$-triple.

We will use the basis

$$\underbrace{\{ H_\alpha \mid \alpha \in \Delta \}}_{\text{basis of } \mathfrak{h}} \cup \underbrace{\{ X_\alpha \mid \alpha \in R \}}_{\text{basis of } \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha}$$

We write H_α' and X_α' for the corresponding elements of the dual basis.

Hence we have

$$C = \sum_{\alpha \in \Delta} H_\alpha H_\alpha' + \sum_{\alpha \in R} X_\alpha X_\alpha'$$

<u>Step 2</u> The action of $\sum_{\alpha \in \Delta} H_\alpha H'_\alpha$ on V_μ

Let $(\omega_\alpha)_{\alpha\in\Delta}$ be the fundamental weights.

Recall that they are a basis for the weight lattice.

Let $(\omega'_\alpha)_{\alpha\in\Delta}$ be the dual basis.

Write $r_\alpha = \mu(H_\alpha)$ and $r'_\alpha = \mu(H'_\alpha)$, so that

$$\mu = \sum r_\alpha \omega_\alpha \text{ and } \mu = \sum r'_\alpha \omega'_\alpha .$$

In particular $(\mu,\mu) = \sum r_\alpha r'_\alpha$.

Clearly $H_\alpha H'_\alpha$ acts on V_μ via multiplication by $\mu(H_\alpha)\mu(H'_\alpha) = r_\alpha r'_\alpha$.

Hence $\sum H_\alpha H'_\alpha$ acts via multiplication by (μ,μ).

<u>Step 2</u> The action of $X_\alpha X'_\alpha$

By properties of the Killing form

$$B(H_\alpha, H_\alpha) = B([X_\alpha, Y_\alpha], H_\alpha) = B(X_\alpha, [Y_\alpha, H_\alpha])$$
$$= B(X_\alpha, \alpha(H_\alpha) Y_\alpha) = 2\, B(X_\alpha, Y_\alpha)$$

We see that

$$B(X_\alpha, Y_\alpha) = B(H_\alpha, H_\alpha)/2$$
$$B(X_\alpha, X'_\alpha) = 1$$

Since $\mathfrak{g}_{-\alpha}$ is 1-dimensional, we conclude

$$X'_\alpha = 2/B(H_\alpha, H_\alpha) \cdot Y_\alpha = (\alpha, \alpha)/2 \cdot Y_\alpha .$$

We want to understand the action on V_μ of

$$X_\alpha X'_\alpha = (\alpha,\alpha)/2 \; X_\alpha Y_\alpha .$$

Now consider $\bigoplus V_{\mu+i\alpha}$ as representation of the sL_2-triple $(X_\alpha, H_\alpha, Y_\alpha)$.

This is a situation that we understand.

Write

$$\bigoplus V_{\mu+i\alpha} = V_{\beta-m\alpha} \oplus \cdots \oplus \underset{\displaystyle V_\mu}{\overset{\shortparallel}{V_{\beta-k\alpha}}} \oplus \cdots \oplus V_{\beta-\alpha} \oplus V_\beta$$

so that $m = \beta(H_\alpha)$.

Now we will recursively split this decomposition into smaller pieces.

On V_β we know that $X_\alpha Y_\alpha$ acts as $m = \beta(H_\alpha) = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$.

Hence $X_\alpha X_\alpha'$ acts as multiplication by (β,α).

Now we use more representation theory of sl_2.

Recall that Y_α^k maps V_β injectively to a subspace of $V_\mu = V_{\beta-k\alpha}$,

and on this subspace $X_\alpha Y_\alpha$ acts by multiplication with $(k+1)(m-k)$.

So we have found a subspace of V_μ of dimension n_β on which

$X_\alpha X_\alpha'$ acts via multiplication by

$$(k+1)\left((\beta,\alpha) - k(\alpha,\alpha)/2\right) = (k+1)\left((\mu,\alpha) + k(\alpha,\alpha)/2\right).$$

Now we "delete" the sl_2-subrepresentation of $\bigoplus V_{\mu + i\alpha}$ generated by V_β, by restricting to a complement.

We get a subspace of V_μ of dimension $n_\mu - n_\beta$ on which we perform the same analysis:

We find a subspace of dimension $n_{\beta-\alpha} - n_\beta$ on which $X_\alpha X_\alpha'$ acts by multiplication with $k\left((\mu,\alpha) + (k-1)(\alpha,\alpha)/2\right)$

Now we recursively repeat this process.

We get the following formula for the trace of $X_\alpha X_\alpha'$ on V_μ.

Assume for the moment that $k \leq m/2$.

$$
\begin{aligned}
\mathrm{Trace}(X_\alpha X'_\alpha \mid V_\mu) = \; & n_\beta \cdot (k+1) \cdot \big((\mu,\alpha) + k \cdot (\alpha,\alpha)/2\big) \\
& + (n_\beta - n_{\beta-\alpha}) \cdot k \cdot \big((\mu,\alpha) + (k-1)(\alpha,\alpha)/2\big) \\
& + \dots \\
& + (n_{\beta-k\alpha} - n_{\beta-(k-1)\alpha}) \cdot 1 \cdot (\mu,\alpha) + 0 \cdot (\alpha,\alpha)/2\big)
\end{aligned}
$$

Cancelling the terms in this telescoping sum we find

$$\mathrm{Trace}(X_\alpha X'_\alpha \mid V_\mu) = \sum_{i=0}^{k} (\mu + i\alpha, \alpha)\, n_{\mu+i\alpha}$$

Here we use $k \leq m/2$.

In the other case we have to start from the other end of the string

$$V_{\beta-m\alpha} \oplus \dots \oplus V_\mu \oplus \dots \oplus V_\beta$$

Since $n_{\mu+i\alpha} = 0$ for $i > k$, we may just as well consider the infinite sum

$$\mathrm{Trace}(X_\alpha X_\alpha' \mid V_\mu) = \sum_{i \geq 0} (\mu + i\alpha, \alpha)\, n_{\mu+i\alpha}$$

As indicated, if $k \geq m/2$, we should have started our analysis from the other end of the string.

Alternatively, we can use that σ_α is an involution on this string and exploit this symmetry.

In the end, we get for $k \geq m/2$

$$\mathrm{Trace}(X_\alpha X_\alpha' \mid V_\mu) = -\sum_{i<0} (\mu + i\alpha, \alpha)\, n_{\mu+i\alpha}.$$

Of course this case distinction is a bit awkward.

Luckily we can get rid of it.

Subclaim: $\sum_{i\geq 0} (\mu+i\alpha,\alpha)\, n_{\mu+i\alpha} = -\sum_{i<0} (\mu+i\alpha,\alpha)\, n_{\mu+i\alpha}$

Indeed, the symmetry σ_α and $m = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ give

$$(\beta-i\alpha,\alpha)\, n_{\beta-i\alpha} + (\beta-(m-i)\alpha,\alpha)\, n_{\beta-(m-i)\alpha} = (2\beta-m\alpha,\alpha)\, n_{\beta-i\alpha}$$
$$= 0.$$

Hence $\sum_{i\in\mathbb{Z}} (\mu+i\alpha,\alpha)\, n_{\mu+i\alpha} = 0$, which proves the subclaim.

We conclude $\operatorname{Trace}(X_\alpha X'_\alpha \mid V_\mu) = \sum_{i\geq 0} (\mu+i\alpha,\alpha)\, n_{\mu+i\alpha}$.

Step 3 Adding up the pieces.

We find $\mathrm{Trace}(C \mid V_\mu) = (\mu,\mu) n_\mu + \sum_{\alpha \in R} \sum_{i \geq 0} (\mu + i\alpha, \alpha) n_{\mu + i\alpha}$.

Since V is irreducible, every $\mathfrak{g}$-equivariant operator acts as multiplication by a scalar.

In other words, there is some c such that $C(v) = c \cdot v$ for all $v \in V$.

Hence $\mathrm{Trace}(C \mid V_\mu) = c \cdot n_\mu$.

$$c \cdot n_\mu = (\mu,\mu) n_\mu + \sum_{\alpha \in R} \sum_{i \geq 0} (\mu + i\alpha, \alpha) n_{\mu + i\alpha}.$$

Now we are almost done.

$$c \cdot n_\mu = (\mu,\mu) n_\mu + \sum_{\alpha \in R} \sum_{i \geq 0} (\mu + i\alpha, \alpha) n_{\mu+i\alpha} .$$

If $i = 0$ then the two terms for α and $-\alpha$ cancel. So we get

$$c \cdot n_\mu = (\mu,\mu) n_\mu + \sum_{\alpha \in R} \sum_{i \geq 1} (\mu + i\alpha, \alpha) n_{\mu+i\alpha} .$$

Now split the double sum into a sum over the positive roots

$$\sum_{\alpha \in R} \sum_{i \geq 1} (\mu + i\alpha, \alpha) n_{\mu+i\alpha}$$

$$= \sum_{\alpha \in R^+} \sum_{i \geq 1} (\mu + i\alpha, \alpha) n_{\mu+i\alpha} + \sum_{\alpha \in R^+} \sum_{i \geq 1} (\mu - i\alpha, -\alpha) n_{\mu-i\alpha}$$

$$= \sum_{\alpha \in R^+} \sum_{i \geq 1} (\mu + i\alpha, \alpha) n_{\mu+i\alpha} + \sum_{\alpha \in R^+} - \sum_{i \geq 1} (\mu - i\alpha, \alpha) n_{\mu-i\alpha}$$

continued on next page.

$$= \sum_{\alpha \in R^+} \sum_{i \geq 1} (\mu + i\alpha, \alpha)\, n_{\mu + i\alpha} \quad + \quad \sum_{\alpha \in R^+} - \sum_{i \geq 1} (\mu - i\alpha, \alpha)\, n_{\mu - i\alpha}$$

$$= \sum_{\alpha \in R^+} \sum_{i \geq 1} (\mu + i\alpha, \alpha)\, n_{\mu + i\alpha} \quad + \quad \sum_{\alpha \in R^+} \sum_{i \leq 0} (\mu - i\alpha, \alpha)\, n_{\mu - i\alpha} \qquad \text{(by the subclaim)}$$

$$= \sum_{\alpha \in R^+} \sum_{i \geq 1} (\mu + i\alpha, \alpha)\, n_{\mu + i\alpha} \quad + \quad \sum_{\alpha \in R^+} \sum_{i \geq 0} (\mu + i\alpha, \alpha)\, n_{\mu + i\alpha}$$

$$= \sum_{\alpha \in R^+}' (\mu, \alpha)\, n_\mu \quad + \quad 2 \sum_{\alpha \in R^+} \sum_{i \geq 1} (\mu + i\alpha, \alpha)\, n_{\mu + i\alpha}$$

Now note that $\displaystyle \sum_{\alpha \in R^+}' (\mu, \alpha) = (\mu, 2\delta)$.

So we find

$$c \cdot n_\mu = \big((\mu, \mu) + (\mu, 2\delta) \big) \cdot n_\mu \quad + \quad 2 \sum_{\alpha \in R^+} \sum_{i \geq 1} (\mu + i\alpha, \alpha)\, n_{\mu + i\alpha}$$

Finally, we determine c. Consider $\mu = \lambda$.

In this case $n_\lambda = 1$ and $n_{\lambda+i\alpha} = 0$ for $i > 0$.

Thus the formula from the previous page gives

$$c = (\lambda,\lambda) + (\lambda, 2\delta).$$

Altogether, for arbitrary μ, we find

$$\big((\lambda,\lambda) + (\lambda,2\delta) - (\mu,\mu) - (\mu,2\delta)\big)\cdot n_\mu = 2\sum_{\alpha\in R^+}\sum_{i\geq 1}(\mu+i\alpha,\alpha)\, n_{\mu+i\alpha}$$

Now observe that $c(\mu) = \|\lambda+\delta\|^2 - \|\mu+\delta\|^2 = (\lambda+\delta, \lambda+\delta) - (\mu+\delta, \mu+\delta)$

$$= (\lambda,\lambda) + (\lambda,2\delta) - (\mu,\mu) - (\mu,2\delta)$$

as desired. This finishes the proof of Freudenthal's formula $\blacksquare$