Existence of semisimple representations

Last time:

- every standard cyclic module of weight λ has a unique irreducible quotient $V(\lambda)$
- two irreducible standard cyclic modules of weight λ are isomorphic

This time:

- for every λ there exists a standard cyclic module of weight λ
- when is the irreducible quotient $V(\lambda)$ finite-dimensional?

<u>Setup</u> As usual:

K	algebraically closed, char. 0 field
$\mathfrak{g}$	finite-dimensional semisimple Lie algebra over K
$\mathfrak{h} \subset \mathfrak{g}$	Cartan subalgebra
R	root system of $\mathfrak{h} \subset \mathfrak{g}$
Δ	a base of R.

Fix a $\lambda \in \mathfrak{h}^*$. We want to construct a standard cyclic module of weight λ, that we will denote $Z(\lambda)$.

For every positive root $\alpha > 0$, choose $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $(X_\alpha, H_\alpha, Y_\alpha)$ is an $\mathfrak{sl}_2$-triple.

Now consider $U(\mathfrak{g})$, the universal enveloping algebra, and let $I(\lambda)$ be the ideal generated by

$$\{X_\alpha \mid \alpha \in R^+\} \cup \{H_\alpha - \lambda(H_\alpha) \cdot 1 \mid \alpha \in R\}$$

Now define $Z(\lambda)$ to be $U(\mathfrak{g})/I(\lambda)$.

$Z(\lambda)$ is a module over $U(\mathfrak{g})$ and hence a representation of $\mathfrak{g}$ via the natural map $\mathfrak{g} \to U(\mathfrak{g})$.

Let v^+ be the image of 1 in the quotient $Z(\lambda)$.

By construction: $X_\alpha(v^+) = 0$ for all $\alpha \in R^+$

$H_\alpha(v^+) = \lambda(H_\alpha) \cdot v^+$ for all $\alpha \in R$

Hence we conclude:

<u>Theorem</u> $Z(\lambda)$ is a standard cyclic module of weight λ ∎

Definition The Verma module of weight λ, denoted $V(\lambda)$, is the unique irreducible quotient of $Z(\lambda)$.

A corollary from what we have seen before:

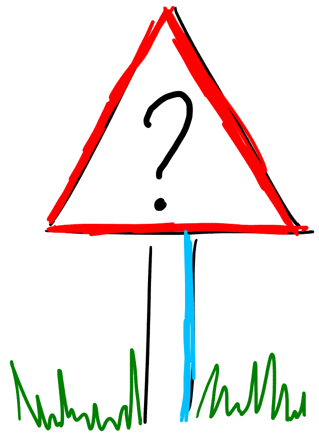
Every irreducible standard cyclic module of weight λ is isomorphic to $V(\lambda)$.

Let V be a finite-dimensional irreducible representation of $\mathfrak{g}$.

- then it must have a maximal vector v^+ of some weight λ (by finite-dimensionality)

- v^+ generates a submodule of V, which must be all of V itself (by irreducibility)

We conclude that V is isomorphic to $V(\lambda)$.



The question remains:

For which $\lambda \in \mathfrak{h}^*$ is $V(\lambda)$ finite-dimensional ?

Here we will connect to the weight theory of the root system R that we looked at last week.

Necessary conditions

Suppose that $V = V(\lambda)$ is finite-dimensional.

Let $S_\alpha = (X_\alpha, H_\alpha, Y_\alpha)$ be an $\mathfrak{sl}_2$-triple for each $\alpha \in \Delta$.

Then V is a finite-dimensional representation of S_α.

Since λ is a highest weight of V as representation of $\mathfrak{g}$, we can consider $\lambda|_{\langle H_\alpha \rangle} \in \langle H_\alpha \rangle^*$, and notice that it is the highest weight of V as representation of $S_\alpha \cong \mathfrak{sl}_2$.

But $\dim \langle H_\alpha \rangle^* = 1$, and we identified weights of sl_2 with their image on $H = H_\alpha$.

So in this case, the highest weight of V as representation of S_α is $\lambda(H_\alpha)$.

In particular we know that $\lambda(H_\alpha)$ is a non-negative integer, by our understanding of the representation theory of sl_2.

More generally, if V is any finite-dimensional representation of $\mathfrak{g}$ and μ a weight of $\mathfrak{g}$, then $\mu(H_\alpha) = \langle \mu, \alpha \rangle \in \mathbb{Z}$ for $\alpha \in \Delta$.

Hence weights of V are weights of the root system R.

We found a necessary condition for $\dim V(\lambda) < \infty$

λ must be a dominant weight of the root system R

Is this condition also sufficient? (Hint: yes)

For a representation V, let $P(V)$ denote its set of weights.
For $V = V(\lambda)$ we write $P(\lambda) = P(V)$.

<u>Theorem</u> If $\lambda \in \mathfrak{h}^*$ is a dominant weight of R, then $V(\lambda)$ is finite-dimensional, and $P(\lambda)$ is a saturated set of weights.
For $\sigma \in W$ and $\mu \in P(\lambda)$ we have $\dim V(\lambda)_\mu = \dim V(\lambda)_{\sigma(\mu)}$.

<u>Corollary</u> The map $\lambda \mapsto V(\lambda)$ gives a 1-to-1 correspondence between the set Λ^+ of dominant weights and isomorphism classes of semisimple finite-dimensional representations of $\mathfrak{g}$.

We will prove the theorem in several steps, after proving a helper lemma.

__Helper lemma__ (Compare with what we did for $\mathfrak{sl}_3$)

In $\mathcal{U}(\mathfrak{g})$ we have

(i) $[X_j, Y_i^{k+1}] = 0$ for $i \neq j$

(ii) $[H_j, Y_i^{k+1}] = -(k+1)\, \alpha_i(H_j)\, Y_i^{k+1}$

(iii) $[X_i, Y_i^{k+1}] = -(k+1)\, Y_i^{k}(k \cdot 1 - H_i)$

__Proof__ (i) Use that $\alpha_j - \alpha_i$ (more precisely $\alpha_j - (k+1)\alpha_i$) is not a root.

(ii) By induction on k. $k=0$ is clear. Now we repeat the fundemental calculation

$$[H_j, Y_i^{k+1}] = H_j Y_i^{k+1} + Y_i^{k+1} H_j = (H_j Y_i^k - Y_i^k H_j) Y_i + Y_i^k [H_j, Y_i]$$

$$= -k\,\alpha_i(H_j)\, Y_i^k Y_i + Y_i^k \alpha_i(H_j) Y_i = -(k+1)\, \alpha_i(H_j)\, Y_i^{k+1}$$

(iii) Calculate using induction on k

$$[X_i, Y_i^{k+1}] = X_i Y_i^{k+1} - Y_i^{k+1} X_i = [X_i, Y_i] Y_i^k + Y_i [X_i, Y_i^k]$$

$$= H_i Y_i^k + Y_i \left(-k Y_i^{k-1}\left((k-1)\cdot 1 - H_i\right)\right)$$

$$= H_i Y_i^k - k(k-1) Y_i^k + (k+1) Y_i^k H_i - Y_i^k H_i$$

$$= [H_i, Y_i^k] - k(k-1) Y_i^k + (k+1) Y_i^k H_i$$

$$\overset{(ii)}{=} -k \, \alpha_i(H_i) Y_i^k - k(k-1) Y_i^k + (k+1) Y_i^k H_i$$

$$= -k \cdot 2 \cdot Y_i^k - k(k-1) Y_i^k + (k+1) Y_i^k H_i$$

$$= -k(k+1) Y_i^k + (k+1) Y_i^k H_i$$

$$= -(k+1) Y_i^k (k - H_i) \qquad \blacksquare$$

Proof of the theorem

As before, let $S_\alpha = (X_\alpha, H_\alpha, Y_\alpha)$ be an sl_2-triple for $\alpha \in \Delta$.

Fix a maximal vector v^+ of weight λ, and write $m_\alpha = \lambda(H_\alpha) \in \mathbb{Z}_{\geq 0}$ for $\alpha \in \Delta$.

Step 1 $Y_\alpha^{m_\alpha+1} v^+ = 0$. Write $w = Y_\alpha^{m_\alpha+1} v^+$. If $\beta \neq \alpha$ then $X_\beta w = 0$ by (i) of the lemma. By (ii) and (iii) of the lemma,

$$X_\alpha w = X_\alpha Y_\alpha^{m_\alpha+1} v^+ = [X_\alpha, Y_\alpha^{m_\alpha+1}] v^+ + Y_\alpha^{m_\alpha+1} X_\alpha v^+$$

$$= Y_\alpha^{m_\alpha+1} X_\alpha v^+ - (m_\alpha+1) Y_\alpha^{m_\alpha} \cdot (m_\alpha v^+ - H_\alpha v^+)$$

$$= 0 - (m_\alpha+1) Y_\alpha^{m_\alpha} \cdot (m_\alpha v^+ - \underbrace{\lambda(H_\alpha)}_{= m_\alpha} v^+) = 0$$

If $w \neq 0$, this shows that it is a maximal vector of weight $\lambda - (m_\alpha+1)\alpha$. But $V(\lambda)$ is irreducible, and hence the maximal weight is unique. Hence $w = 0$.

Step 2 For $\alpha \in \Delta$, there is a finite-dimensional S_α-subrepresentation of V.

Take the S_α-subrepresentation generated by v^+.

It is the linear span of $v^+, Y_\alpha v^+, \ldots, Y_\alpha^i v^+$.

By step 1 this is finite-dimensional.

Step 3 V is the sum of finite-dimensional S_i-submodules.

Let V' be the sum of all such submodules. By step 2, $V' \neq 0$.

If W is a finite-dimensional S_i-submodule, then the span of all $X_\alpha W$ $(\alpha \in R)$

is finite-dimensional and S_i-stable. Hence V' is $\mathfrak{g}$-stable.

By irreducibility of $V = V(\lambda)$, we conclude $V' = V$.

Step 4 For all $v \in V$, X_α and Y_α act nilpotently on v.

Indeed, v lies in a finite sum of findim S_α-submodules, by step 3, hence in a findim S_α-submodule. On that submodule, X_α and Y_α act nilpotently by the representation theory of findim $\mathfrak{sl}_2$-representations.

Step 5 Form the automorphism $s_\alpha(v) = \exp(X_\alpha)\exp(-Y_\alpha)\exp(X_\alpha)(v)$

Here $\exp(\phi) = \sum_{n \geq 0} \frac{1}{n!}\phi^n$. Note that $s_\alpha(v)$ is well-defined because of step 4.

Step 6 If μ is any weight of V, then $s_\alpha(V_\mu) = V_{\sigma_\alpha \mu}$.

We omit this calculation from the proof.

See the calculation that we did for $\mathfrak{sl}_3$ for the strategy.

The upshot is that $P(\lambda)$ is stable under the action of $\mathcal{W}$.

Step 7 $P(\lambda)$ is $\mathcal{W}$-stable, and $\dim V_\mu = \dim V_{\sigma\mu}$ for $\mu \in P(\lambda)$, $\sigma \in \mathcal{W}$.

Immediate from step 6.

Step 8 $P(\lambda)$ is finite.

Every weight is $\mathcal{W}$-conjugate to a dominant weight.

Now use step 7 and the fact that $P(\lambda)$ only contains dominant weights μ with $\mu \leq \lambda$ of which there are finitely many.

Step 9 V is finite-dimensional

In the previous lecture we saw that V_μ is finite-dimensional.

Now use step 8.

Step 10 $P(\lambda)$ is saturated.

The space $\bigoplus_{i \in \mathbb{Z}} V_{\mu + i\alpha}$ is an S_α-subrepresentation.

Hence the α-string through μ must be uninterrupted.

We also know that σ_α reverses the string.

If the string is $\mu - r\alpha, \ldots, \mu, \ldots, \mu + q\alpha$, then we have seen that $r - q = \langle \mu, \alpha \rangle$.

Hence $P(\lambda)$ is saturated. $\blacksquare$

This finishes the classification of finite-dimensional representations of $\mathfrak{g}$.

Next time: Can we calculate $\dim V_\mu$?