# Constructing root systems and Lie algebras

Goal: For each connected Dynkin diagram construct a corresponding irreducible root system and simple Lie algebra.

This is the "existence" part of the classification theorem.

Strategy [Sketch!] Pick a Dynkin diagram with $\ell$ vertices.

Inside the standard Euclidean space $\mathbb{R}^n$ (for a suitable $n$)

with standard orthonormal basis $e_1, \ldots, e_n$

find an $\ell$-dimensional subspace $E$, and consider

the lattice $\Lambda \subset E$ of vectors with integer coefficients (on the standard basis).

Let $R \subset E$ be the set of vectors in $\Lambda$ with certain prescribed lengths.

We must then check that this results in a root system with

the correct Dynkin diagram.

This must mostly be done case by case.

- We will exclude $0$ from $R$ by definition.
- Since $\Lambda$ is discrete and $\{v \in E \mid \|v\| \leq c\}$ is compact, we know that $R$ is compact.
- Checking $\langle \alpha, \beta \rangle \in \mathbb{Z}$ and $\sigma_\alpha(R) \subset R$ for $\alpha, \beta \in R$ must be done on individual basis.

I will leave this last step as an exercise.

<u>The case $A_\ell$</u> Consider the subspace $E \subset \mathbb{R}^{\ell+1}$ of vectors orthogonal to $e_1 + \cdots + e_{\ell+1}$. Consider $\Lambda = (\mathbb{Z}e_1 + \cdots + \mathbb{Z}e_{\ell+1}) \cap E$ and let $R \subset \Lambda$ be the set $\{\alpha \in \Lambda \mid (\alpha,\alpha) = 2\}$.

Note that $R = \{e_i - e_j \mid i \neq j\}$.

Let $\Delta \subset R$ be the set of vectors $\alpha_i = e_i - e_{i+1}$ $(i = 1, \ldots, \ell)$.

Then $\Delta$ is an independent set.

Note that $e_i - e_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ for $i<j$. Hence $\Delta$ is a base for $R$.

The reflection $\sigma_{\alpha_i}$ permutes $e_i$ and $e_j$ and fixes the other basis vectors.
So it corresponds to the transposition $(i, i+1)$ in the symmetric group $S_{\ell+1}$ and we conclude that $\mathcal{W} \cong S_{\ell+1}$.

**The case $B_\ell$** Take $E = \mathbb{R}^\ell$ and $R \subset E$ the set

$$\{ \alpha \in \Lambda \mid (\alpha, \alpha) = 1 \text{ or } 2 \}.$$

Then $R$ is the set containing exactly

- the short roots $\pm e_i$
- the long roots $\pm(e_i + e_j)$ and $\pm(e_i - e_j)$ for $i \neq j$

Take $$\Delta = \underbrace{\{ e_1 - e_2, e_2 - e_3, \ldots, e_{\ell-1} - e_\ell \}}_{\text{long roots}} \cup \{ e_\ell \}.$$ short root

Note that $e_i = (e_i - e_{i+1}) + (e_{i+1} - e_{i+2}) + \ldots + (e_{\ell-1} - e_\ell) + e_\ell$

and $e_i + e_j = (e_i - e_{i+1}) + (e_{i+1} - e_{i+2}) + \ldots + 2(e_j - e_{j+1}) + \ldots + 2(e_{\ell-1} - e_\ell) + 2e_\ell$

$e_i - e_j = (e_i - e_{i+1}) + \ldots + (e_{j-1} - e_j).$

So $\Delta$ is a base of $R$.

$\mathcal{W}$ acts on $\{e_1, \ldots, e_n\}$ via permutations (as with $A_\ell$)

but also via sign changes $e_i \mapsto -e_i$ via $\sigma_{e_i}$.

So $\mathcal{W}$ is isomorphic to the semidirect product of $(\mathbb{Z}/2\mathbb{Z})^\ell$ and $S_\ell$,

where $S_\ell$ acts on $(\mathbb{Z}/2\mathbb{Z})^\ell$ via permutations.

<u>The case $C_\ell$</u> Since $C_\ell$ is dual to $B_\ell$ we can construct it as such.

Take $E = \mathbb{R}^\ell$ and let $R$ be the set of all $\pm 2e_i$, $\pm(e_i + e_j)$, and $\pm(e_i - e_j)$, $i \neq j$.

The base is
$$\Delta = \underbrace{\{e_1 - e_2, e_2 - e_3, \ldots, e_{\ell-1} - e_\ell\}}_{\text{short roots}} \cup \underset{\text{long root}}{\{2e_\ell\}}$$

Since dual root systems have isomorphic Weyl groups, we also know

the Weyl group of $C_\ell$.

The case $D_\ell$ Once again, take $E = \mathbb{R}^\ell$.

For $R$, we take the set $\{\alpha \in \Lambda \mid (\alpha, \alpha) = 2\}$, which is the set of all $e_i \pm e_j$ and $-(e_i \pm e_j)$ for $i \neq j$.

The base consists of

$$\Delta = \{e_1 - e_2, e_2 - e_3, \dots, e_{\ell-1} - e_\ell, \quad e_{\ell-1} + e_\ell\}.$$

This time, the Weyl group consists of permutations and simultaneous sign changes that change an even number of signs.

Hence $\mathcal{W}$ is isomorphic to a semidirect product of $(\mathbb{Z}/2\mathbb{Z})^{\ell-1}$ and $S_\ell$.

$E_6, E_7, E_8$ will be done last.

<u>The case $F_4$</u> Take $E = \mathbb{R}^4$. This time we use a slightly different $\Lambda$.

Put $\Lambda = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 + \mathbb{Z}e_4 + \mathbb{Z}(e_1+e_2+e_3+e_4)/2$.

Now let $R$ be $\{\alpha \in \Lambda \mid (\alpha,\alpha) = 1 \text{ or } 2\}$.

Then $R$ contains

$$\left.\begin{array}{l} \square \quad \pm e_i \\ \square \quad (\pm e_1 \pm e_2 \pm e_3 \pm e_4)/2 \end{array}\right\} \text{short roots}$$

$$\left.\square \quad \pm e_i \pm e_j, \; i \neq j \right\} \text{long roots}$$

where all signs may be chosen independently.

As base, take $\Delta = \{\underbrace{e_2 - e_3, e_3 - e_4}_{\text{long}}, \underbrace{e_4, (e_1 - e_2 - e_3 - e_4)/2}_{\text{short}}\}$.

The Weyl group turns out to have order 1152.

**The case $G_2$** We have already seen $G_2$ as example of a root system of rank 2. It was described explicitly in the exercises.

**The case $E_\ell$** If $R \subset E$ is a root system with base $\Delta$, then subsets of $\Delta$ span sub-root systems.

So it suffices to construct $E_8$.

This is somewhat complicated. We start with $E = \mathbb{R}^8$.

Take $\Lambda'' = \mathbb{Z}e_1 + \ldots + \mathbb{Z}e_8$, $\Lambda' = \Lambda'' + \mathbb{Z}(e_1 + \ldots + e_8)/2$ and finally

$$\Lambda = \left\{ \sum n_i e_i + \tfrac{n}{2}(e_1 + \ldots + e_8) \in \Lambda' \;\middle|\; \sum n_i \text{ is even} \right\}.$$

**Exercise** Check that $\Lambda$ is a well-defined subgroup of $E$.

The set of roots is $R = \{ \alpha \in \Lambda \mid (\alpha, \alpha) = 2 \}$.

Then $R$ contains

- $\pm e_i \pm e_j$, $i \neq j$, independent signs.
- $\frac{1}{2} \sum (-1)^{k_i} e_i$, $k_i \in \{0,1\}$, $\sum k_i$ is even.

<u>Exercise</u> Check $(\alpha, \beta) \in \mathbb{Z}$ for all $\alpha, \beta \in R$, hence $\langle \alpha, \beta \rangle \in \mathbb{Z}$.

As base we take

$$\Delta = \{ \tfrac{1}{2}(e_1 - (e_2 + \dots + e_7) - e_8),\ e_1 + e_2,\ e_2 - e_1,\ e_3 - e_2,\ e_4 - e_3,\ e_5 - e_4,\ e_6 - e_5,\ e_7 - e_6 \}.$$

With respect to this ordering we get the following Cartan matrix.

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \end{pmatrix}$$

<u>Exercise</u> Check this. Check that this Cartan matrix gives the correct Dynkin diagram.

Fun fact: $\mathcal{W}$ has $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$ elements.

Now that we have constructed root systems out of Dynkin diagrams, our next goal is to construct a Lie algebra out of a root system.

Let $K$ be a field of characteristic $0$, and let $X$ be a set.

We will now describe the free Lie algebra on $X$.

- ☐ Let $V$ be a $K$-vector space with basis $X$.
- ☐ Recall the tensor algebra $T(V) = \bigoplus_i V^{\otimes i}$.
- ☐ Endow $T(V)$ with the commutator bracket.
- ☐ Let $L$ be the sub-Lie algebra of $T(V)$ generated by $X$.

Universal property of the free Lie algebra.

<u>Theorem</u> Let $\mathfrak{g}$ be another Lie algebra, and $\phi : X \to \mathfrak{g}$ any function. Then there is a unique Lie algebra homomorphism $\psi : L \to \mathfrak{g}$ such that $\psi(x) = \phi(x)$ for all $x \in X$.

<u>Proof</u> Omitted. The proof is not completely obvious, but very formal.

Let $R \subset E$ be a root system, with base $\Delta = (\alpha_1, \dots, \alpha_\ell)$.

Let $L$ be the free Lie algebra with generators

$$\{ X_i, Y_i, H_i \mid i = 1, \dots, \ell \}$$

Let $I \subset L$ be the Lie ideal generated by

- $[H_i, H_j]$
- $[X_i, Y_i] - H_i$, $\quad [X_i, Y_j]$ for $i \neq j$
- $[H_i, X_j] - \langle \alpha_j, \alpha_i \rangle X_j$, $\quad [H_i, Y_j] + \langle \alpha_j, \alpha_i \rangle Y_j$
- $(\mathrm{ad}\ X_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(X_j)$
- $(\mathrm{ad}\ Y_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(Y_j)$

**Theorem** (Serre) The quotient $\mathfrak{g} = L/I$ is a finite-dimensional semisimple Lie algebra with root system $R$.

**Proof** Omitted. This proof is somewhat long and technical. It will be more interesting to look at representations in the final weeks.