EXERCISES FOR THE COURSE "COXETER GROUPS AND LIE ALGEBRAS"

1. Week 1 — discussion on 10 Nov

1.1. **Exercise.** Let K be a field, and let A be a K-algebra, not necessarily associative. A *derivation* is a linear map $D\colon A \to A$ satisfying the Leibniz rule $D(ab) = aD(b) + D(a)b$.

(i) Show that the derivations $\mathrm{Der}(A)$ form a Lie algebra under the bracket $[D, E] = D \circ E - E \circ D$. If A is finite-dimensional, so is $\mathrm{Der}(A)$.

(ii) If the algebra A is a Lie algebra, the map $A \to \mathrm{Der}(A)$, $X \mapsto D_X$, where $D_X(Y) = [X, Y]$, is a map of Lie algebras.

1.2. **Exercise.** Let A be an associative K-algebra. Denote by $\mathfrak{g}$ the Lie algebra $(A, [\cdot, \cdot])$ attached to A. Show that every derivation of A is a derivation of $\mathfrak{g}$. Show that the converse fails. (Hint for the 2nd part: Assume that A is commutative and consider the identity.)

1.3. **Exercise.** Let K be a field, and let $M_n(K)$ denote the K-algebra of $n \times n$-matrices. Let E_{ij} denote the matrix all of whose coefficients are 0 except for the coefficient (i, j), which is 1. In particular, $(E_{ij})_{ij}$ is the standard basis for $M_n(K)$.

(i) Verify the following identities.

$$\begin{cases} [E_{ij}, E_{kl}] = 0 & \text{if } j \neq k \text{ and } i \neq l \\ [E_{ij}, E_{jl}] = E_{il} & \text{if } i \neq l \\ [E_{ij}, E_{ki}] = -E_{kj} & \text{if } j \neq k \\ [E_{ij}, E_{ji}] = E_{ii} - E_{jj} & \end{cases}$$

(ii) Verify that the following sub-vector spaces are sub-Lie algebras of $M_n(K)$.

(a) the subspace of matrices with trace 0;
(b) the subspace of upper-triangular matrices;
(c) the subspace of strict upper-triangular matrices (i.e., with 0 on the diagonal).

1.4. **Exercise** (not for presentation). Let x_1, x_2, x_3, x_4 be elements of a Lie algebra. Show

$$[[[x_1, x_2], x_3], x_4] + [[[x_2, x_1], x_4], x_3] + [[[x_3, x_4], x_1], x_2] + [[[x_4, x_3], x_2], x_1] = 0.$$

2. Week 2 — discussion on 17 Nov

2.1. **Exercise.** Show that $\mathfrak{g}$ is nilpotent if and only if $\mathrm{ad}(X)$ is a nilpotent endomorphism of $\mathfrak{g}$ for every $X \in \mathfrak{g}$.

2.2. **Exercise.** (*i*) Show that up to isomorphism there is 1 Lie algebra of dimension 1. Is it abelian/nilpotent/solvable?

(*ii*) Show that up to isomorphism there are 2 Lie algebras of dimension 2. For both of them, determine whether they are abelian/nilpotent/solvable/simple.

2.3. **Exercise.** Show that every finite-dimensional irreducible representation V of a solvable Lie algebra $\mathfrak{g}$ has dimension 1, and that $\mathfrak{g}$ acts trivially on V.

2.4. **Exercise.** Let $\mathfrak{g}$ be a *simple* Lie algebra. Show that $\mathfrak{g}$ is *perfect*: $\mathcal{D}\mathfrak{g} = \mathfrak{g}$.

2.5. **Exercise.** In the lecture we saw that a Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form is nondegenerate.

Deduce that every semisimple Lie algebra is a direct sum of simple Lie algebras.

2.6. **Exercise.** If $\mathfrak{g}$ is a direct sum of simple Lie algebra $\mathfrak{g}_1, \ldots \mathfrak{g}_n$, show that every ideal of $\mathfrak{g}$ is a direct sum of the $\mathfrak{g}_i$.

Conclude that the decomposition of a semisimple Lie algebra into simple subalgebras is unique (not just up to isomorphism).

3. Week 3 — discussion on 24 Nov

3.1. **Exercise.** Let V be a representation of a Lie algebra $\mathfrak{g}$ over a field K. The *dual* representation $V^* = \mathrm{Hom}(V, K)$ is defined by

$$X(f)(v) = -f(X(v)), \qquad \text{for } X \in \mathfrak{g},\ f \in V^*, \text{ and } v \in V$$

Check that this law satisfies the axioms for a representation.

3.2. **Exercise.** Let V be a finite-dimensional irreducible representation of a Lie algebra $\mathfrak{g}$.

(i) Show that that V^* is irreducible as well.
(ii) If $\mathfrak{g}$ is simple of dimension 3, conclude that $V^* \cong V$ are representations of $\mathfrak{g}$.

3.3. **Exercise.** Let V be a finite-dimensional vector space over an algebraically closed field K. Let $f, g \colon V \to V$ be two *semisimple* endomorphisms. Assume that f and g commute: $f \circ g = g \circ f$.

Show that V admits a basis consisting of vectors that are eigenvectors for both f and g.

3.4. **Exercise** (Functoriality of the Jordan decomposition)**.** Let $f \colon V \to W$ be a linear map between finite-dimensional vector spaces over an algebraically closed field K. Let $T \colon V \to V$ and $U \colon W \to W$ be two endomorphisms, and assume that $f \circ T = U \circ f$.

Let $T = T_s + T_n$ and $U = U_s + U_n$ be the Jordan decompositions of T and U. Show that $f \circ T_s = U_s \circ f$ and $f \circ T_n = U_n \circ f$.

3.5. **Exercise.** Let K be a perfect field, let V be a finite-dimensional vector space over K, and let $f \colon V \to V$ be an endomorphism. Prove that there exists a unique Jordan decomposition of f. (Hint: use Galois theory.)

4. Week 4 — discussion on 01 Dec

4.1. **Exercise** (not for presentation). (*i*) Let V and W be two finite-dimensional vector spaces. Construct a natural isomorphism

$$\mathrm{Hom}(V, W) = V^* \otimes W.$$

(*ii*) Let L/K be a field extension, and let V be a finite-dimensional K-vector space. Show that $V \otimes L$ (tensor product of K-vector spaces) is naturally an L-vector space. Also, show $\dim_L(V \otimes L) = \dim_K(V)$.

4.2. **Exercise.** Let V be a finite-dimensional vector space over a field K. Let $\mathrm{Sym}^*(V)$ denote the symmetric *algebra* $\bigoplus_n \mathrm{Sym}^n(V)$.

(*i*) Show that $\mathrm{Sym}^*(V)$ is naturally a graded K-algebra.

(*ii*) Let $R = \bigoplus_n R_n$ be a commutative graded algebra, and $f\colon V \to R_1$ a K-linear map. Show that f induces a natural graded-algebra homomorphism $\mathrm{Sym}^*(V) \to R$.

(*iii*) Let W be another finite-dimensional K-vector space. Use the preceding point to deduce natural isomorphisms

$$\mathrm{Sym}^*(V \oplus W) \cong \mathrm{Sym}^*(V) \otimes \mathrm{Sym}^*(W)$$

and

$$\mathrm{Sym}^k(V \oplus W) \cong \bigoplus_{i+j=k} \mathrm{Sym}^i(V) \otimes \mathrm{Sym}^j(W).$$

4.3. **Exercise.** In the lecture we saw that the finite-dimensional irreducible representations of $\mathfrak{sl}_2$ are exactly the representations $\mathrm{Sym}^n(V)$, where V is the standard representation of dimension 2.

For $m, n \in \mathbb{N}$, decompose $\mathrm{Sym}^m(V) \otimes \mathrm{Sym}^n(V)$ as sum of irreducible representations.

5. Week 5 — discussion on 08 Dec

In these exercises, *representation* means "finite-dimensional representation of $\mathfrak{sl}_3$".

5.1. **Exercise.** In the lectures, we have seen the *root lattice* Λ_R, and also the *weight lattice* $\Lambda_W \subset \mathfrak{h}^*$ generated by $\{L_1, L_2, L_3\}$.

(*i*) Show that $\Lambda_W/\Lambda_R \cong \mathbb{Z}/3\mathbb{Z}$.
(*ii*) Show that if V is an irreducible representation then the set of weights $P(V)$ of V is a subset of Λ_W.
(*iii*) Show that for every $\alpha \in \Lambda_W$ there exists a representation V with $\alpha \in P(V)$.

5.2. **Exercise.** Let V be the standard representation. For all natural numbers n, show that $\mathrm{Sym}^n(V)$ and $\mathrm{Sym}^n(V^*)$ are irreducible representations.

5.3. **Exercise.** Let V be the standard representation, with standard basis $\{e_1, e_2, e_3\}$. Let $\{e_1^*, e_2^*, e_3^*\}$ be the dual basis of V^*.

In this exercise we study the representation $W = \mathrm{Sym}^2(V) \otimes V^*$.

(*i*) Draw the weight diagram of W.
(*ii*) Show that $\alpha = e_1^2 \otimes e_3^*$ is a heighest weight vector (with respect to the ordering chosen in the lecture).
(*iii*) Let $W' \subset W$ be the subrepresentation generated by α. In the lecture, we saw that the weight space W'_{L_1} is generated by the images of α under successive applications of E_{ij}, with $i > j$. Compute an upper bound on $\dim W'_{L_1}$, and deduce from the weight diagram that W is *not* irreducible.
(*iv*) Explicitly compute the image of α under $E_{2,1} \circ E_{3,2}$ and $E_{3,2} \circ E_{2,1}$. Deduce that $W \cong W' \oplus V$.

6. Week 6 — discussion on 15 Dec

6.1. **Exercise.** In this exercise we show that $\mathfrak{sl}_2$ is isomorphic to $\mathfrak{sp}_2$, over an algebraically closed field K of characteristic 0. Let V be the standard representation of $\mathfrak{sl}_2$.

(i) Compute the dimension of $\bigwedge^2(V)^*$.

(ii) Conclude that there is an alternating $\mathfrak{sl}_2$-invariant bilinear form $q\colon V \times V \to K$. Do you know an example of such an alternating bilinear form?

(iii) Describe how this bilinear form acts on the standard basis of W, and conclude that it is nondegenerate.

(iv) Finally, deduce that the representation $\mathfrak{sl}_2 \to \mathfrak{gl}(W)$ factors via $\mathfrak{sp}(V, q) \cong \mathfrak{sp}_2$, and conclude that $\mathfrak{sl}_2$ and $\mathfrak{sp}_2$ are isomorphic.

6.2. **Exercise.** This exercise is very similar to the first exercise. In this exercise we show that $\mathfrak{sl}_2$ is isomorphic to $\mathfrak{so}_3$, over an algebraically closed field K of characteristic 0. Let V be the standard representation of $\mathfrak{sl}_2$, and let $W = \mathrm{Sym}^2(W)$ be the irreducible 3-dimensional representation of $\mathfrak{sl}_2$.

(i) Draw the weight diagram of $\mathrm{Sym}^2(W)^*$ as representation of $\mathfrak{sl}_2$.

(ii) Conclude that there is a symmetric $\mathfrak{sl}_2$-invariant bilinear form $q\colon W \times W \to K$.

(iii) Describe how this bilinear form acts on the standard basis of W, and conclude that it is nondegenerate.

(iv) Finally, deduce that the representation $\mathfrak{sl}_2 \to \mathfrak{gl}(W)$ factors via $\mathfrak{so}(W, q) \cong \mathfrak{so}_3$, and conclude that $\mathfrak{sl}_2$ and $\mathfrak{so}_3$ are isomorphic.

6.3. **Exercise.** Let $V_1, \ldots, V_n$, and W be representations of a Lie algebra $\mathfrak{g}$. Let $\mathrm{MLin}(V_1, \ldots, V_n; W)$ be the space of multilinear maps $V_1 \times \cdots \times V_n \to W$.

(i) For $X \in \mathfrak{g}$ and $f \in \mathrm{MLin}(V_1, \ldots, V_n; W)$, define $X(f)$ via

$$X(f)(v_1, \ldots, v_n) = X(f(v_1, \ldots, v_n)) - \sum_{i=1}^{n} f(v_1, \ldots, X(v_i), \ldots, v_n).$$

Show that this action of $\mathfrak{g}$ on $\mathrm{MLin}(V_1, \ldots, V_n; W)$ is a representation.

(ii) Show that the representation defined in the preceding point is naturally isomorphic (as representation of $\mathfrak{g}$) to $V_1^* \otimes \cdots \otimes V_n^* \otimes W$.

6.4. **Exercise.** Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra over K. Let $\beta\colon \mathfrak{g} \times \mathfrak{g} \to K$ be an $\mathfrak{g}$-invariant bilinear form on $\mathfrak{g}$. Show that β is symmetric.

7. Week 7 — discussion on 22 Dec

7.1. **Exercise** (not for presentation)**.** Compute the dimensions of the classical Lie algebras $\mathfrak{sl}_n$, $\mathfrak{sp}_{2n}$, and $\mathfrak{so}_n$.

7.2. **Exercise.** Let V be the standard representation of $\mathfrak{sl}_2$. Compute the Casimir operator of W (acting on W) for the following representations W:

(i) the standard representation V,
(ii) the adjoint representation $\mathrm{Sym}^2(V) = \mathfrak{sl}_2$,
(iii) the direct sum of the preceding two representations: $V \oplus \mathfrak{sl}_2$.

(The Casimir operator does not depend on the choice of basis, but of course it helps to pick a basis with nice properties.)

The last exercise may look somewhat out of place, but it is a nice preparation for the material that we will look at after the Christmas break.

7.3. **Exercise.** Let G be a connected graph without loops and without multi-edges. For every node $x \in G$, let $\mathcal{N}_x$ denote the set of *neighbours*: the set of nodes that are adjacent to x.

In this exercise say that G is *good* if there exists a labeling $n\colon G \to \mathbb{R}_{>0}$ of the nodes of G by positive real numbers such that for all $x \in G$ we have

$$2n(x) = \sum_{y \in \mathcal{N}_x} n(y).$$

Classify all good graphs. (There are infinitely many good graphs, but their shapes follow very regular patterns. Describe those patterns.)

8. Week 8 — discussion on 12 Jan

8.1. **Exercise.** Consider the numbers

$$n_{\beta,\alpha} = 2\frac{\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}$$

where α and β range over the simple positive roots (after choosing an ordering of the roots).

In the lecture, we claimed that $n_{\beta,\alpha}$ is an integer. In this exercise we want to consider some of the consequences.

(i) Let ϑ denote the angle between α and β. Show that $n_{\beta,\alpha} = 2\cos(\theta)\frac{\|\beta\|}{\|\alpha\|}$.

(ii) Deduce that $n_{\beta,\alpha}n_{\alpha,\beta} = 4\cos^2(\theta)$.

(iii) For a given pair (α,β) of simple roots, with $\|\alpha\| \leq \|\beta\|$, find all the possible values for:

$$\cos(\theta), \quad \theta, \quad n_{\beta,\alpha}, \quad n_{\alpha,\beta}, \quad \frac{\|\beta\|}{\|\alpha\|}$$

(Hint: part of this data was give in the lecture. You should complete the data, and argue why it is correct, using the preceding parts of this exercise.)

8.2. **Exercise.** The Cartan matrix of a semisimple Lie algebra is the matrix

$$n_{\beta,\alpha} = 2\frac{\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}$$

where α and β range over the simple positive roots (after choosing an ordering of the roots). In the lecture, we constructed the Dynkin diagram from the Cartan matrix.

(i) Show that you can reconstruct the Cartan matrix from the Dynkin diagram.

(ii) Compute the Cartan matrix, and its determinant, for each connected Dynkin diagram.

8.3. **Exercise.** Let H be an element of a finite-dimensional Lie algebra $\mathfrak{g}$ over some field K. Decompose $\mathfrak{g}$ as

$$\mathfrak{g} = \bigoplus_{\lambda\in K} \mathfrak{g}_\lambda(H)$$

where $\mathfrak{g}_\lambda(H) = \{X \in \mathfrak{g} \mid (\mathrm{ad}(H) - \lambda I)^k X = 0 \text{ for some } k\}$.

Show that $[\mathfrak{g}_\lambda(H), \mathfrak{g}_\mu(H)] \subset \mathfrak{g}_{\lambda+\mu}$.

8.4. **Exercise.** Show that the subalgebra $\mathfrak{h} \subset \mathfrak{sl}_4$ of diagonal matrices is a Cartan subalgebra. Compute the root space decomposition of $\mathfrak{sl}_4$. How many roots are there?

Can you draw a picture of the situation? (Hint: it will be 3-dimensional. See also the 2-dimensional picture for $\mathfrak{sl}_3$.)

8.5. **Exercise** (Challenge)**.** This exercise continues the preceding exercise.

Choose an ordering of the roots of $\mathfrak{sl}_4$. Find the set of simple positive roots. Confirm that $\mathfrak{sl}_4$ has Dynkin diagram A_3.

8.6. **Exercise** (Bonus challenge)**.** Perform the preceding to exercises for the Lie algebra $\mathfrak{sp}_4$ or $\mathfrak{so}_n$ $(n = 4, 5)$.

9. Week 9 — discussion on 19 Jan

9.1. **Exercise.** Let E be a euclidean space.

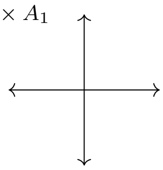
(i) Show that reflections are orthogonal (that is, preserve the inner product).

(ii) For $\alpha \in E$ nonzero, show that $\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ defines a reflection in the hyperplane $P_\alpha = \{\beta \mid (\beta, \alpha) = 0\}$.

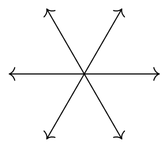
(iii) Let $E' \subset E$ be a subspace. If σ_α leaves E' invariant, prove that either $\alpha \in E'$ or else $E' \subset P_\alpha$.

Examples of root systems of rank 2. (Later, we will see that these are all of them.)

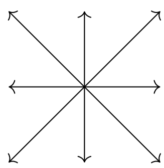
$A_1 \times A_1$



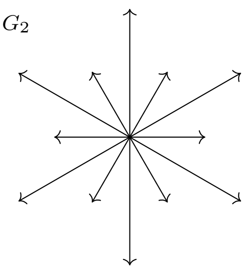
A_2



B_2



G_2



9.2. **Exercise.** Verify the axioms of a root system in the four examples above.

9.3. **Exercise.** Compute the Weyl group of the four root systems depicted above.

9.4. **Exercise.** Let $R \subset E$ be a root system, and let $\alpha, \beta \in R$ be two roots.

(i) If $(\alpha, \beta) > 0$, show that $\alpha - \beta \in R$.

(ii) Is the converse true?

9.5. **Exercise.** Let R be a finite set that spans a euclidean space E. Suppose that all reflections σ_α ($\alpha \in R$) leave R invariant. Let $\sigma : E \to E$ be a linear automorphism that leaves R invariant, fixes some hyperplane $P \subset E$ pointwise, and sends some $\alpha \in R$ to its negative.

In this exercise we will show $\sigma = \sigma_\alpha$ and $P = P_\alpha$.

(i) Show that $\tau = \sigma\sigma_\alpha$ acts as the identity on $\mathbb{R}\alpha$ and $E/\mathbb{R}\alpha$. Deduce that all eigenvalues of τ are equal to 1.

(ii) Show that some power of τ must be the identity, since it preserves the finite set R.

(iii) Deduce that the minimal polynomial of τ must divide $(X-1)^m$ and $X^n - 1$, for some m and n.

(iv) Conclude that τ is the identity, and deduce $\sigma = \sigma_\alpha$ and $P = P_\alpha$.

10. Week 10 — discussion on 26 Jan

10.1. **Exercise** (not for presentation)**.** Prove that $R^\vee$ is a root system in E, whose Weyl group is naturally isomorphic to $\mathcal{W}$. Also show that $\langle \alpha^\vee, \beta^\vee \rangle = \langle \beta, \alpha \rangle$.

10.2. **Exercise.** Let $R' \subset R$ be a subset such that $R' = -R'$, and such that for $\alpha, \beta \in R'$, if $\alpha + \beta \in R$, then $\alpha + \beta \in R'$. Let $E' \subset E$ be the sub-vector space generated by R'.

Show that R' is a root system in E'.

10.3. **Exercise.** Let c be a positive real number, let $R' \subset R$ be subset of roots of length c, and let E' be the subspace of E generated by R'.

Show that R' is a root system in E'.

10.4. **Exercise** (not for presentation)**.** Compute root strings in G_2 to verify the relation $r - q = \langle \beta, \alpha \rangle$.

10.5. **Exercise** (not for presentation)**.** For each of the root systems of rank 2, depicted above:

- (i) Draw the Weyl chambers.
- (ii) Describe the action of the Weyl group on the set of Weyl chambers.
- (iii) Pick a Weyl chamber, and construct the corresponding base of the root system.
- (iv) Choose a simple root α (that is, a root in the base you constructed above). Verify that σ_α permutes the set of positive roots not equal to α.

10.6. **Exercise** (not for presentation)**.** Let $R^\vee$ be the dual root system of R, and let $\Delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta\}$. Prove that $\Delta^\vee$ is a base of $R^\vee$.

10.7. **Exercise.** Prove that every root system of rank 2 is isomorphic to one of $A_1 \times A_1$, A_2, B_2, or G_2. (See the picture on sheet 9.)

11. Week 11 — discussion on 02 Feb

11.1. **Exercise.** Work out the algorithm mentioned in the lecture on the classification of root systems: It should take as input a Cartan matrix of a root system, and output the list of roots as $\mathbb{Z}$-linear combinations of the simple roots.

11.2. **Exercise** (not for presentation)**.** Run the algorithm from the previous exercise on some small Cartan matrices, for example those of G_2, A_3, B_3, C_3.

11.3. **Exercise.** Show that the root systems B_ℓ and C_ℓ are dual to each other. Show that all other root systems are self-dual.

11.4. **Exercise.** Show that the Weyl group of a root system is isomorphic to the direct product of the Weyl groups of its irreducible components.

11.5. **Exercise** (not for presentation)**.** Compute the Weyl groups of the root systems of rank 2. Compute the Weyl group of A_3.

12. Week 12 — discussion on 09 Feb

12.1. **Exercise.** Give an example (they exist with rank 2) of a root system R, with base Δ, simple root $\alpha \in \Delta$, and weight $\lambda \in \Lambda_W$, such that

$$\lambda \notin \Lambda_W^+, \qquad \lambda - \alpha \in \Lambda_W^+.$$

12.2. **Exercise** (not for presentation)**.** For the irreducible root systems R of rank 2, fix a base $\Delta = \{\alpha_1, \alpha_2\} \subset R$, and express the fundamental dominant weights λ_1 and λ_2 as $\mathbb{Q}$-linear combinations of the simple roots α_1, and α_2.

Do the same for F_4.

12.3. **Exercise.** For each irreducible root system, compute the determinant of the corresponding Cartan matrix. (This was also part of Exercise 8.2.)

12.4. **Exercise.** Prove that every subset X of Λ_W is contained in a unique smallest saturated set P. Show that P is finite if X is finite.