

EXERCISES FOR THE COURSE
“COXETER GROUPS AND LIE ALGEBRAS”

1. WEEK 1 — DISCUSSION ON 10 NOV

1.1. **Exercise.** Let K be a field, and let A be a K -algebra, not necessarily associative. A *derivation* is a linear map $D: A \rightarrow A$ satisfying the Leibniz rule $D(ab) = aD(b) + D(a)b$.

- (i) Show that the derivations $\text{Der}(A)$ form a Lie algebra under the bracket $[D, E] = D \circ E - E \circ D$. If A is finite-dimensional, so is $\text{Der}(A)$.
- (ii) If the algebra A is a Lie algebra, the map $A \rightarrow \text{Der}(A)$, $X \mapsto D_X$, where $D_X(Y) = [X, Y]$, is a map of Lie algebras.

1.2. **Exercise.** Let A be an associative K -algebra. Denote by \mathfrak{g} the Lie algebra $(A, [\cdot, \cdot])$ attached to A . Show that every derivation of A is a derivation of \mathfrak{g} . Show that the converse fails. (Hint for the 2nd part: Assume that A is commutative and consider the identity.)

1.3. **Exercise.** Let K be a field, and let $M_n(K)$ denote the K -algebra of $n \times n$ -matrices. Let E_{ij} denote the matrix all of whose coefficients are 0 except for the coefficient (i, j) , which is 1. In particular, $(E_{ij})_{ij}$ is the standard basis for $M_n(K)$.

- (i) Verify the following identities.

$$\begin{cases} [E_{ij}, E_{kl}] = 0 & \text{if } j \neq k \text{ and } i \neq l \\ [E_{ij}, E_{jl}] = E_{il} & \text{if } i \neq l \\ [E_{ij}, E_{ki}] = -E_{kj} & \text{if } j \neq k \\ [E_{ij}, E_{ji}] = E_{ii} - E_{jj} \end{cases}$$

- (ii) Verify that the following sub-vector spaces are sub-Lie algebras of $M_n(K)$.
 - (a) the subspace of matrices with trace 0;
 - (b) the subspace of upper-triangular matrices;
 - (c) the subspace of strict upper-triangular matrices (i.e., with 0 on the diagonal).

1.4. **Exercise** (not for presentation). Let x_1, x_2, x_3, x_4 be elements of a Lie algebra. Show

$$[[[x_1, x_2], x_3], x_4] + [[[x_2, x_1], x_4], x_3] + [[[x_3, x_4], x_1], x_2] + [[[x_4, x_3], x_2], x_1] = 0.$$

2. WEEK 2 — DISCUSSION ON 17 NOV

- 2.1. **Exercise.** Show that \mathfrak{g} is nilpotent if and only if $\text{ad}(X)$ is a nilpotent endomorphism of \mathfrak{g} for every $X \in \mathfrak{g}$.
- 2.2. **Exercise.** (i) Show that up to isomorphism there is 1 Lie algebra of dimension 1. Is it abelian/nilpotent/solvable?
(ii) Show that up to isomorphism there are 2 Lie algebras of dimension 2. For both of them, determine whether they are abelian/nilpotent/solvable/simple.
- 2.3. **Exercise.** Show that every finite-dimensional irreducible representation V of a solvable Lie algebra \mathfrak{g} has dimension 1, and that \mathfrak{g} acts trivially on V .
- 2.4. **Exercise.** Let \mathfrak{g} be a *simple* Lie algebra. Show that \mathfrak{g} is *perfect*: $\mathcal{D}\mathfrak{g} = \mathfrak{g}$.
- 2.5. **Exercise.** In the lecture we saw that a Lie algebra \mathfrak{g} is semisimple if and only if its Killing form is nondegenerate.
Deduce that every semisimple Lie algebra is a direct sum of simple Lie algebras.
- 2.6. **Exercise.** If \mathfrak{g} is a direct sum of simple Lie algebra $\mathfrak{g}_1, \dots, \mathfrak{g}_n$, show that every ideal of \mathfrak{g} is a direct sum of the \mathfrak{g}_i .
Conclude that the decomposition of a semisimple Lie algebra into simple subalgebras is unique (not just up to isomorphism).

3. WEEK 3 — DISCUSSION ON 24 NOV

3.1. **Exercise.** Let V be a representation of a Lie algebra \mathfrak{g} over a field K . The *dual* representation $V^* = \text{Hom}(V, K)$ is defined by

$$X(f)(v) = -f(X(v)), \quad \text{for } X \in \mathfrak{g}, f \in V^*, \text{ and } v \in V$$

Check that this law satisfies the axioms for a representation.

3.2. **Exercise.** Let V be a finite-dimensional irreducible representation of a Lie algebra \mathfrak{g} .

(i) Show that V^* is irreducible as well.

(ii) If \mathfrak{g} is simple of dimension 3, conclude that $V^* \cong V$ are representations of \mathfrak{g} .

3.3. **Exercise.** Let V be a finite-dimensional vector space over an algebraically closed field K . Let $f, g: V \rightarrow V$ be two *semisimple* endomorphisms. Assume that f and g commute: $f \circ g = g \circ f$.

Show that V admits a basis consisting of vectors that are eigenvectors for both f and g .

3.4. **Exercise** (Functoriality of the Jordan decomposition). Let $f: V \rightarrow W$ be a linear map between finite-dimensional vector spaces over an algebraically closed field K . Let $T: V \rightarrow V$ and $U: W \rightarrow W$ be two endomorphisms, and assume that $f \circ T = U \circ f$.

Let $T = T_s + T_n$ and $U = U_s + U_n$ be the Jordan decompositions of T and U . Show that $f \circ T_s = U_s \circ f$ and $f \circ T_n = U_n \circ f$.

3.5. **Exercise.** Let K be a perfect field, let V be a finite-dimensional vector space over K , and let $f: V \rightarrow V$ be an endomorphism. Prove that there exists a unique Jordan decomposition of f . (Hint: use Galois theory.)

4. WEEK 4 — DISCUSSION ON 01 DEC

4.1. **Exercise** (not for presentation). (i) Let V and W be two finite-dimensional vector spaces. Construct a natural isomorphism

$$\mathrm{Hom}(V, W) = V^* \otimes W.$$

(ii) Let L/K be a field extension, and let V be a finite-dimensional K -vector space. Show that $V \otimes L$ (tensor product of K -vector spaces) is naturally an L -vector space. Also, show $\dim_L(V \otimes L) = \dim_K(V)$.

4.2. **Exercise.** Let V be a finite-dimensional vector space over a field K . Let $\mathrm{Sym}^*(V)$ denote the symmetric algebra $\bigoplus_n \mathrm{Sym}^n(V)$.

- (i) Show that $\mathrm{Sym}^*(V)$ is naturally a graded K -algebra.
(ii) Let $R = \bigoplus_n R_n$ be a commutative graded algebra, and $f: V \rightarrow R_1$ a K -linear map. Show that f induces a natural graded-algebra homomorphism $\mathrm{Sym}^*(V) \rightarrow R$.
(iii) Let W be another finite-dimensional K -vector space. Use the preceding point to deduce natural isomorphisms

$$\mathrm{Sym}^*(V \oplus W) \cong \mathrm{Sym}^*(V) \otimes \mathrm{Sym}^*(W)$$

and

$$\mathrm{Sym}^k(V \oplus W) \cong \bigoplus_{i+j=k} \mathrm{Sym}^i(V) \otimes \mathrm{Sym}^j(W).$$

4.3. **Exercise.** In the lecture we saw that the finite-dimensional irreducible representations of \mathfrak{sl}_2 are exactly the representations $\mathrm{Sym}^n(V)$, where V is the standard representation of dimension 2.

For $m, n \in \mathbb{N}$, decompose $\mathrm{Sym}^m(V) \otimes \mathrm{Sym}^n(V)$ as sum of irreducible representations.

5. WEEK 5 — DISCUSSION ON 08 DEC

In these exercises, *representation* means “finite-dimensional representation of \mathfrak{sl}_3 ”.

5.1. **Exercise.** In the lectures, we have seen the *root lattice* Λ_R , and also the *weight lattice* $\Lambda_W \subset \mathfrak{h}^*$ generated by $\{L_1, L_2, L_3\}$.

- (i) Show that $\Lambda_W/\Lambda_R \cong \mathbb{Z}/3\mathbb{Z}$.
- (ii) Show that if V is an irreducible representation then the set of weights $P(V)$ of V is a subset of Λ_W .
- (iii) Show that for every $\alpha \in \Lambda_W$ there exists a representation V with $\alpha \in P(V)$.

5.2. **Exercise.** Let V be the standard representation. For all natural numbers n , show that $\text{Sym}^n(V)$ and $\text{Sym}^n(V^*)$ are irreducible representations.

5.3. **Exercise.** Let V be the standard representation, with standard basis $\{e_1, e_2, e_3\}$. Let $\{e_1^*, e_2^*, e_3^*\}$ be the dual basis of V^* .

In this exercise we study the representation $W = \text{Sym}^2(V) \otimes V^*$.

- (i) Draw the weight diagram of W .
- (ii) Show that $\alpha = e_1^2 \otimes e_3^*$ is a highest weight vector (with respect to the ordering chosen in the lecture).
- (iii) Let $W' \subset W$ be the subrepresentation generated by α . In the lecture, we saw that the weight space W'_{L_1} is generated by the images of α under successive applications of E_{ij} , with $i > j$. Compute an upper bound on $\dim W'_{L_1}$, and deduce from the weight diagram that W is *not* irreducible.
- (iv) Explicitly compute the image of α under $E_{2,1} \circ E_{3,2}$ and $E_{3,2} \circ E_{2,1}$. Deduce that $W \cong W' \oplus V$.

6. WEEK 6 — DISCUSSION ON 15 DEC

6.1. Exercise. In this exercise we show that \mathfrak{sl}_2 is isomorphic to \mathfrak{sp}_2 , over an algebraically closed field K of characteristic 0. Let V be the standard representation of \mathfrak{sl}_2 .

- (i) Compute the dimension of $\bigwedge^2(V)^*$.
- (ii) Conclude that there is an alternating \mathfrak{sl}_2 -invariant bilinear form $q: V \times V \rightarrow K$. Do you know an example of such an alternating bilinear form?
- (iii) Describe how this bilinear form acts on the standard basis of W , and conclude that it is nondegenerate.
- (iv) Finally, deduce that the representation $\mathfrak{sl}_2 \rightarrow \mathfrak{gl}(W)$ factors via $\mathfrak{sp}(V, q) \cong \mathfrak{sp}_2$, and conclude that \mathfrak{sl}_2 and \mathfrak{sp}_2 are isomorphic.

6.2. Exercise. This exercise is very similar to the first exercise. In this exercise we show that \mathfrak{sl}_2 is isomorphic to \mathfrak{so}_3 , over an algebraically closed field K of characteristic 0. Let V be the standard representation of \mathfrak{sl}_2 , and let $W = \text{Sym}^2(W)$ be the irreducible 3-dimensional representation of \mathfrak{sl}_2 .

- (i) Draw the weight diagram of $\text{Sym}^2(W)^*$ as representation of \mathfrak{sl}_2 .
- (ii) Conclude that there is a symmetric \mathfrak{sl}_2 -invariant bilinear form $q: W \times W \rightarrow K$.
- (iii) Describe how this bilinear form acts on the standard basis of W , and conclude that it is nondegenerate.
- (iv) Finally, deduce that the representation $\mathfrak{sl}_2 \rightarrow \mathfrak{gl}(W)$ factors via $\mathfrak{so}(W, q) \cong \mathfrak{so}_3$, and conclude that \mathfrak{sl}_2 and \mathfrak{so}_3 are isomorphic.

6.3. Exercise. Let V_1, \dots, V_n , and W be representations of a Lie algebra \mathfrak{g} . Let $\text{MLin}(V_1, \dots, V_n; W)$ be the space of multilinear maps $V_1 \times \dots \times V_n \rightarrow W$.

- (i) For $X \in \mathfrak{g}$ and $f \in \text{MLin}(V_1, \dots, V_n; W)$, define $X(f)$ via

$$X(f)(v_1, \dots, v_n) = X(f(v_1, \dots, v_n)) - \sum_{i=1}^n f(v_1, \dots, X(v_i), \dots, v_n).$$

Show that this action of \mathfrak{g} on $\text{MLin}(V_1, \dots, V_n; W)$ is a representation.

- (ii) Show that the representation defined in the preceding point is naturally isomorphic (as representation of \mathfrak{g}) to $V_1^* \otimes \dots \otimes V_n^* \otimes W$.

6.4. Exercise. Let \mathfrak{g} be a semisimple finite-dimensional Lie algebra over K . Let $\beta: \mathfrak{g} \times \mathfrak{g} \rightarrow K$ be an \mathfrak{g} -invariant bilinear form on \mathfrak{g} . Show that β is symmetric.

7. WEEK 7 — DISCUSSION ON 22 DEC

7.1. **Exercise** (not for presentation). Compute the dimensions of the classical Lie algebras \mathfrak{sl}_n , \mathfrak{sp}_{2n} , and \mathfrak{so}_n .

7.2. **Exercise.** Let V be the standard representation of \mathfrak{sl}_2 . Compute the Casimir operator of W (acting on W) for the following representations W :

- (i) the standard representation V ,
- (ii) the adjoint representation $\mathrm{Sym}^2(V) = \mathfrak{sl}_2$,
- (iii) the direct sum of the preceding two representations: $V \oplus \mathfrak{sl}_2$.

(The Casimir operator does not depend on the choice of basis, but of course it helps to pick a basis with nice properties.)

The last exercise may look somewhat out of place, but it is a nice preparation for the material that we will look at after the Christmas break.

7.3. **Exercise.** Let G be a connected graph without loops and without multi-edges. For every node $x \in G$, let \mathcal{N}_x denote the set of *neighbours*: the set of nodes that are adjacent to x .

In this exercise say that G is *good* if there exists a labeling $n: G \rightarrow \mathbb{R}_{>0}$ of the nodes of G by positive real numbers such that for all $x \in G$ we have

$$2n(x) = \sum_{y \in \mathcal{N}_x} n(y).$$

Classify all good graphs. (There are infinitely many good graphs, but their shapes follow very regular patterns. Describe those patterns.)

8. WEEK 8 — DISCUSSION ON 12 JAN

8.1. **Exercise.** Consider the numbers

$$n_{\beta,\alpha} = 2 \frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle}$$

where α and β range over the simple positive roots (after choosing an ordering of the roots).

In the lecture, we claimed that $n_{\beta,\alpha}$ is an integer. In this exercise we want to consider some of the consequences.

- (i) Let ϑ denote the angle between α and β . Show that $n_{\beta,\alpha} = 2 \cos(\vartheta) \frac{\|\beta\|}{\|\alpha\|}$.
- (ii) Deduce that $n_{\beta,\alpha} n_{\alpha,\beta} = 4 \cos^2(\vartheta)$.
- (iii) For a given pair (α, β) of simple roots, with $\|\alpha\| \leq \|\beta\|$, find all the possible values for:

$$\cos(\vartheta), \quad \vartheta, \quad n_{\beta,\alpha}, \quad n_{\alpha,\beta}, \quad \frac{\|\beta\|}{\|\alpha\|}$$

(Hint: part of this data was give in the lecture. You should complete the data, and argue why it is correct, using the preceding parts of this exercise.)

8.2. **Exercise.** The Cartan matrix of a semisimple Lie algebra is the matrix

$$n_{\beta,\alpha} = 2 \frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle}$$

where α and β range over the simple positive roots (after choosing an ordering of the roots). In the lecture, we constructed the Dynkin diagram from the Cartan matrix.

- (i) Show that you can reconstruct the Cartan matrix from the Dynkin diagram.
- (ii) Compute the Cartan matrix, and its determinant, for each connected Dynkin diagram.

8.3. **Exercise.** Let H be an element of a finite-dimensional Lie algebra \mathfrak{g} over some field K . Decompose \mathfrak{g} as

$$\mathfrak{g} = \bigoplus_{\lambda \in K} \mathfrak{g}_\lambda(H)$$

where $\mathfrak{g}_\lambda(H) = \{X \in \mathfrak{g} \mid (\text{ad}(H) - \lambda I)^k X = 0 \text{ for some } k\}$.

Show that $[\mathfrak{g}_\lambda(H), \mathfrak{g}_\mu(H)] \subset \mathfrak{g}_{\lambda+\mu}$.

8.4. **Exercise.** Show that the subalgebra $\mathfrak{h} \subset \mathfrak{sl}_4$ of diagonal matrices is a Cartan subalgebra. Compute the root space decomposition of \mathfrak{sl}_4 . How many roots are there?

Can you draw a picture of the situation? (Hint: it will be 3-dimensional. See also the 2-dimensional picture for \mathfrak{sl}_3 .)

8.5. **Exercise (Challenge).** This exercise continues the preceding exercise.

Choose an ordering of the roots of \mathfrak{sl}_4 . Find the set of simple positive roots. Confirm that \mathfrak{sl}_4 has Dynkin diagram A_3 .

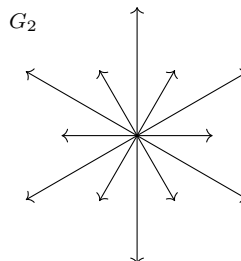
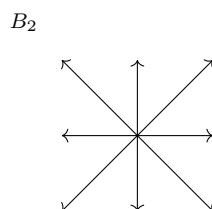
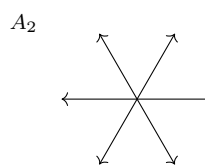
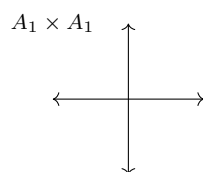
8.6. **Exercise (Bonus challenge).** Perform the preceding to exercises for the Lie algebra \mathfrak{sp}_4 or \mathfrak{so}_n ($n = 4, 5$).

9. WEEK 9 — DISCUSSION ON 19 JAN

9.1. **Exercise.** Let E be a euclidean space.

- (i) Show that reflections are orthogonal (that is, preserve the inner product).
- (ii) For $\alpha \in E$ nonzero, show that $\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha$ defines a reflection in the hyperplane $P_\alpha = \{\beta \mid (\beta, \alpha) = 0\}$.
- (iii) Let $E' \subset E$ be a subspace. If σ_α leaves E' invariant, prove that either $\alpha \in E'$ or else $E' \subset P_\alpha$.

Examples of root systems of rank 2. (Later, we will see that these are all of them.)



9.2. **Exercise.** Verify the axioms of a root system in the four examples above.

9.3. **Exercise.** Compute the Weyl group of the four root systems depicted above.

9.4. **Exercise.** Let $R \subset E$ be a root system, and let $\alpha, \beta \in R$ be two roots.

- (i) If $(\alpha, \beta) > 0$, show that $\alpha - \beta \in R$.
- (ii) Is the converse true?

9.5. **Exercise.** Let R be a finite set that spans a euclidean space E . Suppose that all reflections σ_α ($\alpha \in R$) leave R invariant. Let $\sigma: E \rightarrow E$ be a linear automorphism that leaves R invariant, fixes some hyperplane $P \subset E$ pointwise, and sends some $\alpha \in R$ to its negative.

In this exercise we will show $\sigma = \sigma_\alpha$ and $P = P_\alpha$.

- (i) Show that $\tau = \sigma\sigma_\alpha$ acts as the identity on $\mathbb{R}\alpha$ and $E/\mathbb{R}\alpha$. Deduce that all eigenvalues of τ are equal to 1.
- (ii) Show that some power of τ must be the identity, since it preserves the finite set R .
- (iii) Deduce that the minimal polynomial of τ must divide $(X-1)^m$ and X^n-1 , for some m and n .
- (iv) Conclude that τ is the identity, and deduce $\sigma = \sigma_\alpha$ and $P = P_\alpha$.

10. WEEK 10 — DISCUSSION ON 26 JAN

10.1. **Exercise** (not for presentation). Prove that R^\vee is a root system in E , whose Weyl group is naturally isomorphic to \mathcal{W} . Also show that $\langle \alpha^\vee, \beta^\vee \rangle = \langle \beta, \alpha \rangle$.

10.2. **Exercise**. Let $R' \subset R$ be a subset such that $R' = -R'$, and such that for $\alpha, \beta \in R$, if $\alpha + \beta \in R$, then $\alpha + \beta \in R'$. Let $E' \subset E$ be the sub-vector space generated by R' .

Show that R' is a root system in E' .

10.3. **Exercise**. Let c be a positive real number, let $R' \subset R$ be subset of roots of length c , and let E' be the subspace of E generated by R' .

Show that R' is a root system in E' .

10.4. **Exercise** (not for presentation). Compute root strings in G_2 to verify the relation $r - q = \langle \beta, \alpha \rangle$.

10.5. **Exercise** (not for presentation). For each of the root systems of rank 2, depicted above:

- (i) Draw the Weyl chambers.
- (ii) Describe the action of the Weyl group on the set of Weyl chambers.
- (iii) Pick a Weyl chamber, and construct the corresponding base of the root system.
- (iv) Choose a simple root α (that is, a root in the base you constructed above). Verify that σ_α permutes the set of positive roots not equal to α .

10.6. **Exercise** (not for presentation). Let R^\vee be the dual root system of R , and let $\Delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta\}$. Prove that Δ^\vee is a base of R^\vee .

10.7. **Exercise**. Prove that every root system of rank 2 is isomorphic to one of $A_1 \times A_1$, A_2 , B_2 , or G_2 . (See the picture on sheet 9.)