Engel's theorem

Let V be a representation of a finite-dimensional Lie algebra $\mathfrak{g}$. Assume that all elements of $\mathfrak{g}$ act as nilpotent endomorphisms of V.

If $V \neq 0$, then there exists a $v \in V$, $v \neq 0$ such that $X(v) = 0$ for all $X \in \mathfrak{g}$.

Lemma

Let X be a nilpotent endomorphism of V.

Then
$$\mathrm{ad}(X) : \mathfrak{gl}(V) \longrightarrow \mathfrak{gl}(V)$$
$$Y \longmapsto [X, Y]$$
is nilpotent.

Proof Observe that $\mathrm{ad}(X)^m$ is a sum of terms of the form $\pm X^i Y X^j$, $i+j=m$

Suppose that $X^k = 0$. Then $\mathrm{ad}(X)^{2k-1} = 0$. $\square$

Proof of the theorem

By induction on $n = \dim \mathfrak{g}$.

The cases $n=0$ and $n=1$ are trivial. Assume $n \geq 2$.

Assume the statement for all $m < n$.

It suffices to construct an ideal $\mathfrak{h}$ of dimension $n-1$.

(Claim 1)

Pick $Y \in \mathfrak{g}$, $Y \notin \mathfrak{h}$. Then

$$W = \{ w \in V \mid X(w) = 0 \text{ for all } X \in \mathfrak{h} \}$$

is stable under Y. By induction $W \neq 0$.

(Claim 2)

(cntd)

Pick $Y \in \mathfrak{g}$, $Y \notin \mathfrak{h}$. Then

$$W = \{ w \in V \mid X(w) = 0 \text{ for all } X \in \mathfrak{h} \}$$

is stable under Y. By induction $W \neq 0$.

Recall that Y is nilpotent.

Hence there is $w \neq 0$ in W such that $Y(w) = 0$.

By assumption $X(w) = 0$ for all $X \in \mathfrak{h}$.

Recall $\dim \mathfrak{h} = n-1$ and $Y \notin \mathfrak{h}$.

So $X(w) = 0$ for all $X \in \mathfrak{g}$.

Now prove the claims.

Claim 2

(we will treat claim 1 next)

The space

$$W = \{ w \in V \mid X(w) = 0 \text{ for all } X \in \mathfrak{h} \}$$

is stable under Y

Pick any $w \in W$.

Need to show $X(Y(w)) = 0$, for all $X \in \mathfrak{h}$.

Recall: $$\cancel{\underbrace{[X, Y]}_{\in \mathfrak{h}}(w)} = X(Y(w)) - \cancel{Y(\underbrace{X(w)}_{=0})}$$

$= 0$ Hence $X(Y(w)) = 0$ $\square$

Claim 1 Construct an ideal $\mathfrak{h}$ of dimension $n-1$.

Let $\mathfrak{h} \subset \mathfrak{g}$ be any subalgebra with $0 \neq \mathfrak{h} \neq \mathfrak{g}$.

For example $\langle X \rangle$, with $X \neq 0$, for any $X \in \mathfrak{g}$.

Consider $\mathfrak{g}/\mathfrak{h}$ as representation of $\mathfrak{h}$.

By induction there is a nonzero element of $\mathfrak{g}/\mathfrak{h}$ killed by all $X \in \mathfrak{h}$.

In other words: There is a $Y \in \mathfrak{g}$, $Y \notin \mathfrak{h}$ such that $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$.

(cntd)

In other words: There is a $Y \in \mathfrak{g}$, $Y \notin \mathfrak{h}$ such that

$[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$.

But this means that the subspace generated by

$\mathfrak{h}$ and Y is a subalgebra $\mathfrak{h}'$ of $\mathfrak{g}$

and $\mathfrak{h}$ is an ideal of $\mathfrak{h}'$.

Note that $\dim \mathfrak{h}' = \dim \mathfrak{h} + 1$.

By induction, we find an ideal of $\mathfrak{g}$

of dimension $n-1$. $\square$