Representations of $\mathfrak{sl}_2$

Goal

- understand the (finite-dimensional) representations of $\mathfrak{sl}_2$.

Assumptions

- Complete reducibility

  every representation is semisimple: a direct sum of simple ones.

- Preservation of Jordan decomposition

  if $\rho : \mathfrak{sl}_2 \longrightarrow \mathfrak{gl}(V)$ is a representation, then $\rho(X)$ is semisimple/nilpotent if $\mathrm{ad}(X)$ is.

Recall from last time, that every simple Lie algebra $\mathfrak{g}$ of dimension 3 (over an algebraically closed field $K = \bar{K}$ of characteristic 0) admits a basis $\{H, X, Y\}$ with

$$[H, X] = 2X \qquad [H, Y] = -2Y \qquad [X, Y] = H$$

In particular $\mathrm{ad}(H)$ is diagonal with respect to the basis $\{H, X, Y\}$ whereas $\mathrm{ad}(X)$ and $\mathrm{ad}(Y)$ are nilpotent.

Let $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a finite-dimensional irreducible representation.

By "Preservation of Jordan decomposition" we know that $\rho(H)$ is semisimple while $\rho(X)$ and $\rho(Y)$ are nilpotent.

For $\alpha \in K$, let $V_\alpha \subset V$ denote the eigenspace of $\rho(H)$ for the eigenvalue α.

We have $V = \bigoplus_\alpha V_\alpha$.

Suppose that $v \in V_\alpha$. What can we say about $X(v)$ and $Y(v)$?

$$
\begin{aligned}
H(X(v)) &= X(H(v)) + [H, X](v) \\
&= X(\alpha \cdot v) + 2 \cdot X(v) \\
&= (\alpha + 2) \cdot X(v)
\end{aligned}
$$

Conclusion: $X(v) \in V_{\alpha+2}$ and analogously $Y(v) \in V_{\alpha-2}$.

Suppose that $v \in V_\alpha$. Then

$$H(v) \in V_\alpha, \qquad X(v) \in V_{\alpha+2}, \qquad Y(v) \in V_{\alpha-2}.$$

The subset $A = \{\alpha \in K \mid V_\alpha \neq 0\}$ is finite, because $\dim V < \infty$.

For every α, the subspace $\bigoplus_{n \in \mathbb{Z}} V_{\alpha+2n}$ is stable

under H, X, and Y. Hence it is a subrepresentation.

Conclusion: Since V is irreducible, the set A must be of the form

$$\{\beta, \beta+2, \ldots, \beta+2k\}$$

The set $A = \{\alpha \in K \mid V_\alpha \neq 0\}$ is of the form $\{\beta, \beta+2, \dots, \beta+2k\}$.

Let n denote the last element in this sequence : $\beta + 2k$

$$\begin{tikzcd}
\cdots \arrow[r, bend left, "X"] & V_{n-4} \arrow[l, bend left, "Y"] \arrow[r, bend left, "X"] \arrow[loop below, "H"] & V_{n-2} \arrow[l, bend left, "Y"] \arrow[r, bend left, "X"] \arrow[loop below, "H"] & V_n \arrow[l, bend left, "Y"] \arrow[r, bend left, "X=0"] \arrow[loop below, "H"] & 0
\end{tikzcd}$$

Reminder: $H = [X, Y]$ and $H(v) = \alpha \cdot v$ for $v \in V_\alpha$.

$$
\begin{tikzcd}
\cdots \arrow[r, bend left, "X"] & V_{n-4} \arrow[l, bend left, "Y"] \arrow[r, bend left, "X"] \arrow[loop below, "H"] & V_{n-2} \arrow[l, bend left, "Y"] \arrow[r, bend left, "X"] \arrow[loop below, "H"] & V_n \arrow[l, bend left, "Y"] \arrow[r, bend left, "X=0"] \arrow[loop below, "H"] & 0
\end{tikzcd}
$$

Choose a nonzero element $v \in V_n$.

Claim The linear span of $\{v, Y(v), Y^2(v), \ldots\}$ is stable under Y, H, and X, hence a subrepresentation.

Conclusion: Since V is irreducible, all the V_α are 1-dimensional.

Proof of the claim. Clearly, $\langle v, Y(v), Y^2(v), \dots \rangle$ is stable under Y.

Since $Y^m(v) \in V_{n-2m}$, we see that $H(Y^m(v)) = (n-2m) \cdot Y^m(v)$.

Subclaim $X(Y^m(v)) = m(n-m+1) \cdot Y^{m-1}(v)$

By induction. Case 0: $X(v) = 0$

Case $m+1$:

$$
\begin{aligned}
X(Y^{m+1}(v)) &= Y(X(Y^m(v))) + [X,Y](Y^m(v)) \\
&= Y(m \cdot (n-m+1) \cdot Y^{m-1}(v)) + H(Y^m(v)) \\
&= m \cdot (n-m+1) \cdot Y^m(v) + (n-2m) \cdot Y^m(v) \\
&= (m \cdot (n-m) + m + (n-2m)) \cdot Y^m(v) \\
&= (m+1) \cdot (n-m) \cdot Y^m(v)
\end{aligned}
$$

Summary: $\{ v, Y(v), Y^2(v), \ldots \}$ is a basis of V.

$$H(Y^m(v)) = (n-2m) \cdot Y^m(v)$$
$$X(Y^m(v)) = m \cdot (n-m+1) \cdot Y^{m-1}(v)$$
$$Y(Y^m(v)) = Y^{m+1}(v)$$

<u>Corollary 1</u> Up to isomorphism, V is determined by

$$A = \{ \alpha \in K \mid V_\alpha \neq 0 \}$$

in fact determined by $n \in A$.

Summary: $\{v, Y(v), Y^2(v), \dots\}$ is a basis of V.

$$H(Y^m(v)) = (n-2m) \cdot Y^m(v)$$
$$X(Y^m(v)) = m(n-m+1) \cdot Y^{m-1}(v)$$
$$Y(Y^m(v)) = Y^{m+1}(v)$$

<u>Corollary 2</u> Let m be $\dim V = |A|$.

Then $0 = X(Y^m(v)) = m(n-m+1) \cdot Y^{m-1}(v)$.

Since $Y^{m-1}(v) \neq 0$ and $m \neq 0$ we find $n = m-1$.

In particular $n \in K$ is an integer, and A is symmetric around 0.

__Catalogue__ For every $n \in \mathbb{N}$ there is a unique irreducible representation of dimension $n+1$.

$n = 0$ $\quad V_0 \quad$ $\mathfrak{g}$ acts trivial

$n = 1$

$$\begin{tikzcd} \langle e_2 \rangle \arrow[d, phantom, "\shortparallel"] & \langle e_1 \rangle \arrow[d, phantom, "\shortparallel"] \\ V_{-1} \arrow[r, bend left, "X"] & V_1 \arrow[l, bend left, "Y"] \end{tikzcd}$$

the standard representation $\mathfrak{sl}_2 \subset \mathfrak{gl}(K^2)$

if $\{e_1, e_2\}$ is standard basis of K^2,

then $V_1 = \langle e_1 \rangle$ and $V_{-1} = e_2$

$n = 2$

$$\begin{tikzcd} \langle Y \rangle \arrow[d, phantom, "\shortparallel"] & \langle H \rangle \arrow[d, phantom, "\shortparallel"] & \langle X \rangle \arrow[d, phantom, "\shortparallel"] \\ V_{-2} \arrow[r, bend left, "X"] & V_0 \arrow[l, bend left, "Y"] \arrow[r, bend left, "X"] & V_2 \arrow[l, bend left, "Y"] \end{tikzcd}$$

the adjoint representation $\mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$

also the representation on symmetric bilinear forms on K^2.