

Representations of SL_2

Goal □ understand the (finite-dimensional)
representations of SL_2 .

Assumptions □ Complete reducibility
every representation is semisimple:
a direct sum of simple ones.

□ Preservation of Jordan decomposition

if $\rho: SL_2 \rightarrow GL(V)$ is a representation,
then $\rho(X)$ is semisimple/nilpotent if $ad(X)$ is.

Recall from last time, that every simple Lie algebra \mathfrak{g} of dimension 3

(over an algebraically closed field $K = \bar{K}$ of characteristic 0)

admits a basis $\{H, X, Y\}$ with

$$[H, X] = 2X \quad [H, Y] = -2Y \quad [X, Y] = H$$

In particular $\text{ad}(H)$ is diagonal with respect to the basis $\{H, X, Y\}$

whereas $\text{ad}(X)$ and $\text{ad}(Y)$ are nilpotent.

Let $\rho: \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a finite-dimensional irreducible representation.

By "Preservation of Jordan decomposition" we know that

$\rho(H)$ is semisimple while $\rho(X)$ and $\rho(Y)$ are nilpotent.

For $\alpha \in K$, let $V_\alpha \subset V$ denote the eigenspace of $\rho(H)$

for the eigenvalue α .

We have $V = \bigoplus_{\alpha} V_{\alpha}$.

Suppose that $v \in V_\alpha$. What can we say about $X(v)$ and $Y(v)$?

$$H(X(v)) = X(H(v)) + [H, X](v)$$

$$= X(\alpha \cdot v) + 2 \cdot X(v)$$

$$= (\alpha + 2) \cdot X(v)$$

Conclusion: $X(v) \in V_{\alpha+2}$ and analogously $Y(v) \in V_{\alpha-2}$.

Suppose that $v \in V_\alpha$. Then

$$H(v) \in V_\alpha, \quad X(v) \in V_{\alpha+2}, \quad Y(v) \in V_{\alpha-2}.$$

The subset $A = \{\alpha \in K \mid V_\alpha \neq 0\}$ is finite, because $\dim V < \infty$.

For every α , the subspace $\bigoplus_{n \in \mathbb{Z}} V_{\alpha+2n}$ is stable

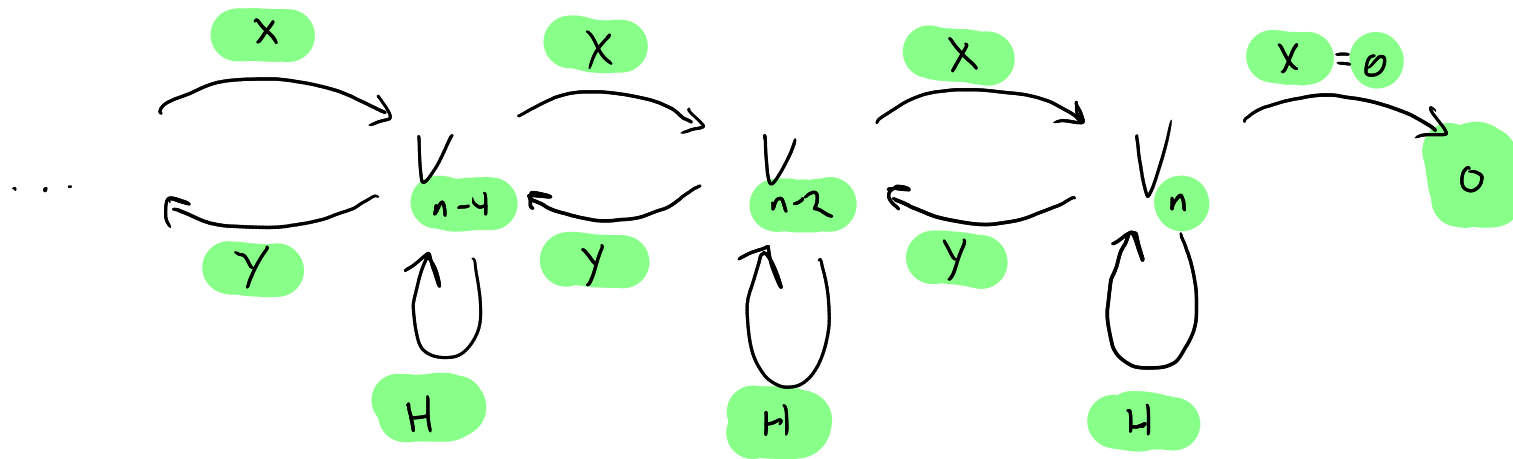
under H , X , and Y . Hence it is a subrepresentation.

Conclusion: Since V is irreducible, the set A must be of the form

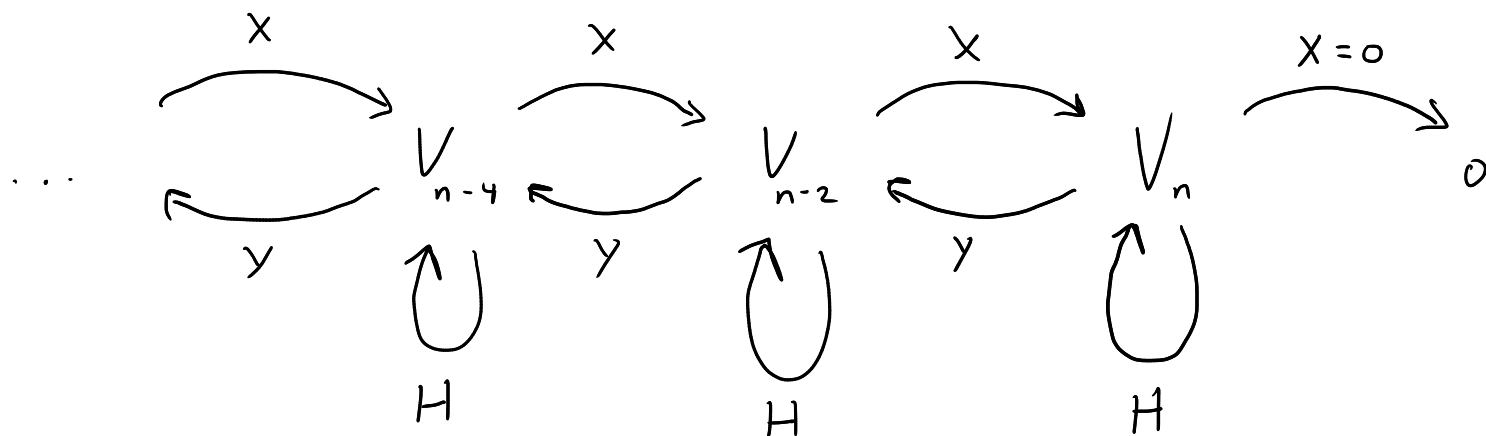
$$\{\beta, \beta+2, \dots, \beta+2k\}$$

The set $A = \{\alpha \in K \mid V_\alpha \neq 0\}$ is of the form $\{\beta, \beta+2, \dots, \beta+2k\}$.

Let n denote the last element in this sequence: $\beta+2k$



Reminder: $H = [X, Y]$ and $H(v) = \alpha \cdot v$ for $v \in V_\alpha$.



Choose a nonzero element $v \in V_n$.

Claim The linear span of $\{v, Y(v), Y^2(v), \dots\}$

is **stable** under Y , H , and X , hence a subrepresentation.

Conclusion: Since V is **irreducible**, all the V_α are **1**-dimensional.

Proof of the claim Clearly, $\langle v, Y(v), Y^2(v), \dots \rangle$ is stable under Y .

Since $Y^m(v) \in V_{n-2m}$, we see that $H(Y^m(v)) = (n-2m) \cdot Y^m(v)$.

Subclaim $X(Y^m(v)) = m(n-m+1) \cdot Y^{m-1}(v)$

By induction. Case 0: $X(v) = 0$

$$\begin{aligned} \text{Case } m+1: \quad X(Y^{m+1}(v)) &= Y(X(Y^m(v))) + [X, Y](Y^m(v)) \\ &= Y(m \cdot (n-m+1) \cdot Y^{m-1}(v)) + H(Y^m(v)) \\ &= m \cdot (n-m+1) \cdot Y^m(v) + (n-2m) \cdot Y^m(v) \\ &= (m \cdot (n-m) + m + (n-2m)) \cdot Y^m(v) \\ &= (m+1) \cdot (n-m) \cdot Y^m(v) \end{aligned}$$

Summary: $\{v, Y(v), Y^2(v), \dots\}$ is a basis of V .

$$H(Y^m(v)) = (n-2m) \cdot Y^m(v)$$

$$X(Y^m(v)) = m \cdot (n-m+1) \cdot Y^{m-1}(v)$$

$$Y(Y^m(v)) = Y^{m+1}(v)$$

Corollary 1 Up to isomorphism, V is determined by

$$A = \{\alpha \in K \mid V_\alpha \neq 0\}$$

in fact determined by $n \in A$.

Summary: $\{v, Y(v), Y^2(v), \dots\}$ is a basis of V .

$$H(Y^m(v)) = (n - 2m) \cdot Y^m(v)$$

$$X(Y^m(v)) = m(n - m + 1) \cdot Y^{m-1}(v)$$

$$Y(Y^m(v)) = Y^{m+1}(v)$$

Corollary 2 Let m be $\dim V = |A|$.

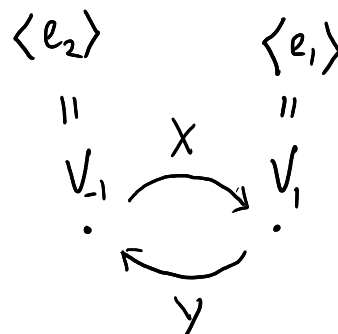
Then $0 = X(Y^m(v)) = m(n - m + 1) \cdot Y^{m-1}(v)$.

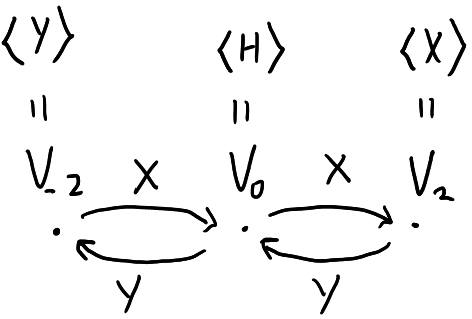
Since $Y^{m-1}(v) \neq 0$ and $m \neq 0$ we find $n = m - 1$.

In particular $n \in K$ is an integer, and A is symmetric around 0 .

Catalogue For every $n \in \mathbb{N}$ there is a unique irreducible representation of dimension $n+1$.

$n = 0$ V_0 \mathfrak{g} acts trivial

$n = 1$  the standard representation $sl_2 \subset \mathfrak{gl}(K^2)$ if $\{e_1, e_2\}$ is standard basis of K^2 , then $V_1 = \langle e_1 \rangle$ and $V_{-1} = \langle e_2 \rangle$

$n = 2$  the adjoint representation $\mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ also the representation on symmetric bilinear forms on K^2 .