Dimension 3

- $\mathfrak{sl}_2$ has dimension 3, it is simple

- over an algebraically closed field $(\operatorname{char} K = 0)$

  it is the unique simple Lie algebra with $\dim = 3$,

  up to isomorphism

Let $\mathfrak{g}$ be a Lie algebra of dimension 3.

$$\dim D\mathfrak{g} = \begin{cases} 0 \rightsquigarrow \mathfrak{g} \text{ abelian} \\ 1 \rightsquigarrow D\mathfrak{g} \text{ abelian}, \ \mathfrak{g} \text{ solvable} \\ 2 \rightsquigarrow D\mathfrak{g} \text{ solvable}, \ \mathfrak{g} \text{ solvable} \\ 3 \rightsquigarrow \mathfrak{g} \text{ simple} \end{cases}$$

Why? If $\mathfrak{h} \subset \mathfrak{g}$ is a proper nontrivial ideal, then $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable. Hence $\mathfrak{g}$ solvable and $D\mathfrak{g} \neq \mathfrak{g}$.

Assume $\dim \mathcal{D}\mathfrak{g} = 3$. In other words $\mathcal{D}\mathfrak{g} = \mathfrak{g}$.

<u>Claim</u> $\operatorname{rk}(\operatorname{ad}(X)) = 2$ for all $0 \neq X \in \mathfrak{g}$.

Extend X to a basis $\{X, Y, Z\}$ of $\mathfrak{g}$.

$\mathcal{D}\mathfrak{g}$ is generated by

$$\{\underset{\substack{\| \\ 0}}{[X,X]}, [X,Y], [X,Z], \underset{\substack{\| \\ 0}}{[Y,Y]}, [Y,Z], \underset{\substack{\| \\ 0}}{[Z,Z]}\}$$

Since $\dim \mathcal{D}\mathfrak{g} = 3$, we conclude that $[X,Y]$ and $[X,Z]$ span a 2-dimensional subspace, so $\operatorname{rk}(\operatorname{ad}(X)) = 2$.

Claim There exists an $H \in \mathfrak{g}$ such that

$$\mathrm{ad}(H) : \mathfrak{g} \longrightarrow \mathfrak{g}$$

has a nonzero eigenvalue.

Proof Let X be any nonzero element of $\mathfrak{g}$.

If $\mathrm{ad}(X)$ has a nonzero eignvalue then we are done.

Otherwise, $\mathrm{ad}(X)$ is nilpotent. [Here we use $K = \overline{K}$.]

Recall that $\ker(\mathrm{ad}(X)) = \langle X \rangle$ and $\dim(\mathrm{im}(\mathrm{ad}(X))) = 2$.

Recall that $\underline{\ker(\mathrm{ad}(X)) = \langle X \rangle \quad \text{and}} \quad \dim(\mathrm{im}(\mathrm{ad}(X))) = 2$.

If $\mathrm{ad}(X)$ is nilpotent, this means that there exists

$$\mathrm{im}(\mathrm{ad}(X)) \cap \ker(\mathrm{ad}(X)) \neq 0$$

More precisely, there exists a $Y \in \mathfrak{g}$ such that

$$\mathrm{ad}(X)(Y) = \alpha X \quad \text{for some } \alpha \neq 0$$

But $\mathrm{ad}(X)(Y) = [X, Y] = -[Y, X] = -\mathrm{ad}(Y)(X)$.

This means that X is an eigenvector of $\mathrm{ad}(Y)$ with eigenvalue $-\alpha \neq 0$. $\square$

Now that the claim is proven, we can choose

$H, X \in \mathfrak{g}$ such that $[H, X] = \mathrm{ad}(H)(X) = \alpha X$ with $\alpha \neq 0$.

Since $D\mathfrak{g} = \mathfrak{g}$, we know $H \in D\mathfrak{g}$, and hence $\mathrm{Tr}(\mathrm{ad}(H)) = 0$.

This means that the eigenvalues of $\mathrm{ad}(H)$ are α, 0, and $-\alpha$.

Let Y be an eigenvector of $\mathrm{ad}(H)$ with eigenvalue $-\alpha$.

Note that $\{X, Y, H\}$ is a basis of $\mathfrak{g}$.

Can we compute $[X, Y]$?

We compute

Jacobi identity

$$[H,[X,Y]] = -[X,[Y,H]] - [Y,[H,X]]$$
$$= -[X, \alpha Y] - [Y, \alpha X]$$
$$= 0$$

Hence $[X,Y] \in \ker(\mathrm{ad}(H)) = \langle H \rangle$.

So there exists a β such that $[X,Y] = \beta H$.

Note that $\beta \neq 0$, for otherwise $Y \in \ker(\mathrm{ad}(X)) = \langle X \rangle$.

Result: $\mathfrak{g}$ is spanned by $\{X, Y, H\}$ and the Lie bracket is

$[-,-]$	H	X	Y
H	0	αX	$-\alpha Y$
X	$-\alpha X$	0	βH
Y	αY	$-\beta H$	0

By scaling H, we can change α to any value in K^*.

By scaling X or Y, we can change β to any value in K^*.

Conclusion: up to isomorphism, there is a unique simple Lie algebra of dimension 3 (over alg. closed fields).

<u>Exercise</u> A natural basis for $\mathfrak{sl}_2$ is

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Check the relations

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H$$

In particular $\mathfrak{sl}_2$ is simple.