Complete reducibility

Theorem (Weyl)

> Every finite-dimensional representation of a semisimple Lie algebra is completely reducible.

(from last time)

<u>Lemma</u> Let $\mathfrak{g}$ be a Lie algebra. The following are equivalent:

(i) every finite-dimensional representation of $\mathfrak{g}$ is semisimple.

(ii) for every finite-dimensional representation V of $\mathfrak{g}$ and every subspace $W \subset V$ of codimension 1 for which $X(V) \subset W$ for all $X \in \mathfrak{g}$, there exists a line $L \subset V$ complementary to W and stable under $\mathfrak{g}$.

Proof strategy By the lemma, it suffices to prove

Let $\mathfrak{g}$ be a semisimple Lie algebra,

V a finite-dimensional representation of $\mathfrak{g}$,

$W \subset V$ a subspace of codimension 1

such that $X(V) \subset W$ for all $X \in \mathfrak{g}$.

Then there is a $\mathfrak{g}$-stable line $L \subset V$

complementary to W.

<u>Step 1</u> Prove the claim if W is irreducible.

<u>Step 2</u> Prove the general claim by induction.

Note that W is a subrepresentation.

If $\mathfrak{g}$ acts trivially on W, then for all $v \in V$, $X, Y \in \mathfrak{g}$ we find $X(Y(v)) = 0$, hence $[X,Y](v) = X(Y(v)) - Y(X(v)) = 0$.

Since $\mathfrak{g} = \mathcal{D}\mathfrak{g}$, this means that $\mathfrak{g}$ acts trivially on V.

In this case we are done.

Now suppose that $\mathfrak{g}$ acts nontrivially on W.

Let $\mathfrak{n} \subset \mathfrak{g}$ be the kernel of the representation $\mathfrak{g} \to \mathfrak{gl}(W)$.

By one of the preparation lemmas, there is an ideal $\mathfrak{h} \subset \mathfrak{g}$ that is complementary to $\mathfrak{n}$.

The representation $\mathfrak{h} \longrightarrow \mathfrak{gl}(W)$ is faithful and $\mathfrak{h}$ is semisimple, hence B_W is a nondegenerate bilinear form on $\mathfrak{h}$.

Let C be the corresponding Casimir operator.

We have seen that C is an automorphism of W,

because in this step W is assumed irreducible.

In addition, we know $C(V) \subset W$.

Hence $L = \mathrm{Ker}(C : V \to W)$ is 1-dimensional.

Since C is $\mathfrak{g}$-invariant, L is a $\mathfrak{g}$-stable line

complementary to W.

We continue with step 2.

Assume that $T \subset W$ is a nontrivial proper subrepresentation.

Now consider the quotient representations $W' \subset V'$, where

$$W' = W/T \quad \text{and} \quad V' = V/T.$$

For all $X \in \mathfrak{g}$ we have $X(V') \subset W'$,

and $W' \subset V'$ has codimension 1.

By induction, there exists a line $L' \subset V'$ that is

$\mathfrak{g}$-stable and complementary to W'.

Let Z be the inverse image of L' in V.

Note that Z is $\mathfrak{g}$-stable, and contains T as codimension 1 subspace.

Since $X(Z) \subset W$ for all X, we find $Z \cap W \subset T$.

By induction, we find a $\mathfrak{g}$-stable line $L \subset Z$ that is complementary to T.

Hence $L \cap W = 0$, and L is also complementary to W.

This means that we win.

$$
\begin{tikzcd}
 & & T \arrow[dll, hook] \arrow[d, hook] \arrow[drr, hook] & & \\
W \arrow[d] & \subset & V \arrow[d] & \supset & Z \arrow[d] \\
W/T & \subset & V/T & \supset & L' \\
\shortparallel & & \shortparallel & & \\
W' & & V' & &
\end{tikzcd}
$$