Complete reducibility (preparations)

Theorem Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over an algebraically closed field of characteristic 0. Let V be a finite-dimensional representation of $\mathfrak{g}$, and let $W \subset V$ be a subrepresentation. Then there exists a subrepresentation $W' \subset V$, such that

$$V = W \oplus W'.$$

Preparation: Schur's lemma. Let $\mathfrak{g}$ be a Lie algebra over $K = \overline{K}$

Let V and W be irreducible representations, and

let $\varphi : V \to W$ be a $\mathfrak{g}$-invariant linear map.

(i) φ is an isomorphism or $\varphi = 0$

(ii) If $V = W$, then $\varphi = \lambda \cdot I$ for some $\lambda \in K$

Proof (i) Note that $\mathrm{Ker}(\varphi)$ and $\mathrm{Im}(\varphi)$ are subrepresentations.

(ii) φ must have an eigenvalue λ. Now $\mathrm{Ker}(\varphi - \lambda I)$ must be a nontrivial subrepresentation, hence all of V. So $\varphi = \lambda I$. $\square$

Lemma Let $\mathfrak{g}$ be a Lie algebra. The following are equivalent:

(i) every finite-dimensional representation of $\mathfrak{g}$ is semisimple.

(ii) for every finite-dimensional representation V of $\mathfrak{g}$ and every subspace $W \subset V$ of codimension 1 for which $X(V) \subset W$ for all $X \in \mathfrak{g}$, there exists a line $L \subset V$ complementary to W and stable under $\mathfrak{g}$.

Proof Clearly (i) $\Rightarrow$ (ii). Assume (ii).

Let V be a finite dimensional representation and $W \subset V$ a subrepresentation.

Let $M \subset \mathrm{End}(V)$ be the subspace of endomorphisms $f: V \to V$ for which $\mathrm{Im}(f) \subset W$ and $f|_W$ is multiplication by a scalar.

Similarly $N = \{ f \in \mathrm{End}(V) \mid \mathrm{Im}(f) \subset W \text{ and } f|_W = 0 \}$.

Remark $N \subset M$ has codimension 1.

By (ii), there exists an $f \in M$ such that f is G-invariant and $f|_W$ is multiplication by some $\lambda \neq 0$. Hence $\lambda^{-1} f$ is a projector from V onto W and since it is G-invariant this means that the kernel of $\lambda^{-1} f$ is a subrepresentation of V that is complementary to W. $\square$

Lemma Let $\mathfrak{h} \subset \mathfrak{g}$ be a semisimple ideal of a finite-dimensional Lie algebra. Then there is an ideal $I \subset \mathfrak{g}$, such that $\mathfrak{g} = \mathfrak{h} \oplus I$

direct sum of vector spaces

Proof Let B be the Killing form of $\mathfrak{g}$. Consider

$$I = \operatorname{Ker}\left(\begin{array}{rcl} \mathfrak{g} & \longrightarrow & \mathfrak{h}^* \\ x & \longmapsto & B(x, \_) \end{array} \right).$$

$Y \in \mathfrak{h}$

Claim I is an ideal. Indeed, suppose that $x, \cancel{\ } \in \mathfrak{g}$ and $i \in I$, then $B([i, x], Y) = B(i, [x, Y]) = 0$, so that $[i, x] \in I$.

Hence $\mathfrak{h} \cap I$ is an ideal. It is semisimple, because $\mathfrak{h}$ is.

Therefore B is non-denerate on $\mathfrak{h} \cap I$, but at the same time

$$B(X, Y) = 0 \quad \text{for } X, Y \in \mathfrak{h} \cap I \subset I.$$

Thus $\mathfrak{h} \cap I = 0$.

Since $I = \operatorname{Ker}(\mathfrak{g} \to \mathfrak{h}^*)$ we conclude

$$\dim \mathfrak{h} + \dim I \leq \dim \mathfrak{g} \leq \dim I + \dim \mathfrak{h}^*$$

and hence $\mathfrak{g} = \mathfrak{h} \oplus I$. $\square$